Coalitional Bargaining with Consistent Counterfactuals*

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Abstract

We propose a new solution concept for TU cooperative games in charateristic function form, the *SCOOP*, that builds on the Nash Bargaining Solution (*NBS*), adding to it a consistency requirement for negotiations inside every coalition. The *SCOOP* specifies the probability of success and payoffs of each coalition. Players share the surplus of a coalition according to the *NBS*, with disagreement payoffs that are computed as the expectation of payoffs in other coalitions, using some common probability distribution, which in turn is derived from the prior distribution. The predicted outcome can be probabilistic or deterministic, but only the efficient coalition can succeed with probability one. We discuss necessary and sufficient conditions for an efficient solution. In either case, the *SCOOP* always exists, is generically unique, easy to compute, and exhibits smooth comparative statics. We also discuss non-cooperative implementation of the *SCOOP*.

KEYWORDS: cooperative games, coalitional bargaining, endogenous disagreement payoffs, consistent beliefs.

JEL classification numbers: C71,C78.

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1 Introduction

The *Nash Bargaining Solution* (*NBS*) has long been accepted as the standard solution concept for negotiations between two —or more— players about how to share the proceeds of an agreement. Less consensus exists as to the right theory for the problem of coalition formation. There, different possible agreements involve different sets of players, each characterized by the worth that the players in the set may obtain and share should they strike that agreement.

We propose a theory for this problem that builds on the *NBS* adding to it a consistency requirement for negotiations inside every coalition. More specifically, we define a solution concept for games in characteristic function form, the *Solution with Consistent Outside OPtions* (*SCOOP*). The *SCOOP* specifies the probability of success as well as payoffs in each coalition. It requires that surplus in every coalition is shared according to the *NBS*, with disagreement points consistently computed. In particular, these disagreement points are the players' expected payoffs in alternative coalitions, using some probability distribution over those counterfactuals, which is shared by all players.[1] Moreover, these assessments are consistent across coalitions. That is, they are conditional distributions obtained from common prior probabilities over all coalitions. Finally, we also require that probabilities are consistent with payoffs: a coalition may be expected to succeed —with positive probability— only if all players involved have no better option.

We begin by showing that the *SCOOP* exists and is —generically— unique for any game, superadditive or not, where only coalitions involving at least half the players have positive worth.

The *SCOOP* may be probabilistic or deterministic, but only the efficient coalition —the one with maximum worth— can succeed with probability one. We characterize the set of games for which the *SCOOP* predicts this efficient outcome. Having a non-empty core is a necessary but not sufficient condition for the existence of an efficient *SCOOP*. A deterministic (efficient) *SCOOP* requires that the payoffs of players in the efficient coalition prefer what they obtain in that coalition over what they might obtain elsewhere. This is also what an imputation must satisfy to be in the core. Hence, in a deterministic *SCOOP* the agreement in the efficient coalition is an allocation in the core. But in addition to this, a deterministic *SCOOP* requires such an agreement to be

[1] We refer to these counterfactuals as outside options, although they are external only when considered from the point of view of negotiations to reach a particular agreement or form a particular coalition.

supported by consistent counterfactuals, a stronger requirement that all core allocations may fail to satisfy. In this case, as in the case of games with an empty core, the *SCOOP* must be probabilistic.

Remarkably, whether its prediction is probabilistic or not, the *SCOOP* is generically unique in probabilities and payoffs. Moreover, its comparative statics with respect to the worth of coalitions are smooth. Thus, the *SCOOP* provides sharp predictions, which, in addition, are relatively straightforward to compute.

Compte and Jehiel (2010) is probably the most immediate predecessor of this paper. There, and in the context of supperaditive games in characteristic function form, the authors define a cooperative solution concept, the Coalition Nash Bargaining Solution (*CNBS*), as the allocation in the core that maximizes the Nash product. They show that a non-cooperative game, the random-proponent protocol, —asymptotically— implements the *CNBS* whenever the —limiting— equilibrium is efficient, and characterize the conditions for this to be the case.

We show that the *SCOOP* coincides with the *CNBS* —a selection of the core— precisely under these conditions The coincidence is not casual: subgame perfection in the non-cooperative protocol implicitly imposes consistency conditions that are related to ours.[2] When the game has a non-empty core but the consistency conditions are not satisfied, then the (probabilistic) *SCOOP* is different both from the limiting equilibria of the random-proponent game and also from the *CNBS*. The *SCOOP* provides, also for these games and games with an empty core, a (generically) unique prediction that exhibits smooth comparative statics.

Our emphasis is not on non-cooperative implementation. However, we discuss why —versions of— the two standard protocols, the random-proponent protocol and also the rejector-proposes protocol, may fail to produce satisfactory predictions.[3] Indeed, exogenous, random —Nature's— moves are less neutral than typically considered, and introduce rigidities in outcomes that are not justified by primitives of the problem. We show how versions of these protocols, that allow for sufficient flexibility in these moves, necessarily identify the *SCOOP* as the equilibrium outcome.

Our paper is also related to Bennett (1997), who formalizes disagreement points in a particular negotiation as the maximum payoffs that players obtain in their alternative negotiations. In particular, in the three-player/three-cake problem (Binmore, 1985) this

[2] See Footnote 5 in Subsection 3.3 for a further discussion of this analogy.

[3] See Ray (2007) for an excellent discussion of these protocols and other related issues.

implies that disagreement payoffs in each bilateral negotiation are the payoffs that each player obtains in their negotiations with the third player. This is equivalent to assuming that players assign probability one to each of the two mutually exclusive counterfactuals. A similar problem arises when we consider various definitions of the reduced-game property that have been invoked to characterize the nucleolus and the Shapley value as appropriate generalizations of the *NBS* (Sobolev, 1975; Peleg, 1986; and Hart and Mas-Colell, 1989). More specifically, disagreement points in the reduced game are defined overlooking the possible incompatibility of alternative agreements.

2 Model

We consider $n > 2$ agents who negotiate to form a coalition or team. The set –and the coalition– of all agents will be represented by N, and the set of coalitions by S. Coalition $s \in S$ has worth $v_s \geq 0$. Thus, the pair $\left(N, \{v_s\}_{s \in S}\right)$ defines a coalitional game: a cooperative game in characteristic function form. Let n_s represent the size of coalition s, $n_s = \#s$. We consider games where $v_s > 0$ only if $n_s > N/2$. Therefore, we concentrate on the problem of forming one coalition. Also, note that we implicitly normalize the worth of all one-player coalitions to 0. Finally, for simplicity, we will assume that $e = \arg\max_s v_s$, the efficient coalition, is unique but does not necessarily coincide with N.

Next, we propose a theory for these negotiations and a solution concept (prediction). That solution should identify an outcome. A *deterministic* outcome is a coalition s, i.e., the identities of agents that form the coalition, and payoffs, u_i^s, for all $i \in s$. We will also allow for non deterministic outcomes. Thus, in general, an outcome is a probability distribution $\mathbf{p}$, over the set of deterministic outcomes.

The elements of our theory are simple: agents will consider all possible negotiations (one in each team $s \in S$) simultaneously. Payoffs in each team must correspond to (generalized) Nash bargaining inside the team, and the disagreement points for each of these negotiations must correspond to the (expected) payoffs in alternative teams. Finally, the expectation is computed using Bayes' rule and a common probability distribution over teams. This probability distribution must be consistent with the agents' payoffs in all teams.

Let $\{p_s\}_{s \in S}$ denote a probability distribution on S. Also, for each s, let $\{\lambda_r^s\}_{r \neq s}$ represent a probability distribution on the set of coalitions minus s. Finally, for each $s \in S$, denote $\{t_i^s\}_{i \in s}$ a vector of disagreement points for the n_s players in coalition s.

Definition 1 *A Solution with Consistent Outside OPtions (SCOOP) for the game* $\left(N, \{v_s\}_{s\in S}\right)$ *is a quadruple* $\sigma = \left\{\{u_i^s\}_{i\in s}, \{t_i^s\}_{i\in s}, p_s, \{\lambda_r^s\}_{r\neq s}\right\}_{s\in S}$ *that satisfies:*

(i) $u_i^s = \begin{cases} t_i^s + \frac{1}{n_s}\left(v_s - \sum_{j\in s} t_j^s\right), & \text{if } v_s \geq \sum_{j\in s} t_j^s \\ 0, & \text{otherwise} \end{cases}$

(ii) $t_i^s = \sum_{r\ni i, r\neq s} \lambda_r^s u_i^r$. If $p_s < 1$ then $\lambda_r^s = \frac{p_r}{1-p_s}$.

(iii) $p_s > 0$ only if $u_i^s \in \max_{r\ni i} u_i^r$, for all $i \in s$.

The first line in (i) is simply the generalized Nash bargaining solution. The second line corresponds to the "outside-option principle": u_i^s may play (part of) the role of an outside option for an agent $i \in s$ in negotiations elsewhere, but only if s is indeed a team that i –and the rest of members of s– may consider joining. Otherwise, payoffs in s must be irrelevant, i.e., add nothing to agents' disagreement points when they negotiate to form other coalitions $r \neq s$.

Part (ii) in the definition establishes that disagreement points in s are the expected payoffs in alternative negotiations $r \neq s$. It also requires that the probability distribution used to calculate these expected payoffs in alternative negotiations is consistent. That is, it must be the conditional probability distribution over coalitions other than s obtained from a common prior $\mathbf{p} = \{p_s\}$ using Bayes' rule, when applicable.

Part (iii) imposes —-strong— consistency of this common prior with payoff predictions. It requires that no player is expected to join a coalitions when she strictly prefers what she gets in an alternative one. Weaker consistency criteria may be defended. Yet, this strong one is not only compatible with existence, but delivers a unique prediction. Weaker criteria would have to include this prediction, and so it may be argued that the *SCOOP* would still be the most robust prediction.

Before discussing the predictions obtained from this concept, we would like to know that the concept always results in a prediction. Our first result guarantees that this is the case.

Theorem 2 *A SCOOP always exists.*

The proof is in the appendix.

3 Analysis

3.1 Characterizing the *SCOOP*

As mentioned in the previous section, a prediction, i.e., a *SCOOP*, may or may not be deterministic. In a deterministic *SCOOP*, $p_r = 1$ for some coalition r, and so $\lambda_s^r = 1$ for all other coalitions $s \neq r$. The next lemma characterizes deterministic *SCOOP*s. In particular, it informs of the coalition that forms, and of the conjectures that sustain the division of surplus. Let us begin by defining, for each σ, the set

$$S_0(\sigma) = \left\{ s \in S \mid p_s = 0 \wedge \sum_{i \in s} t_i^s = v_s \right\},$$

of coalitions that have a worth, net of disagreement points, of exactly zero and for which $p_s = 0$. Also, for each player in e, let

$$\widetilde{q}_i = \sum_{s \ni i, s \in S_0(\sigma)} \lambda_s^e,$$

the probability that i belongs to the coalition that forms conditional on the efficient coalition e not forming, an event that may have probability 0.

Lemma 3 *$\sigma = \{u_i^s, t_i^s, p_s, \lambda_s^r\}$ is a deterministic SCOOP only if: (a) $p_e = 1$. (b) For all $s \in S_0(\sigma) \cup \{e\}$ and $i \in s$, $u_i^s = u_i$ for some u_i, and $\sum_{j \in s \cap e} u_j - v_s = 0$. Moreover, for all s, $u_i^s = 0$ if $i \notin e$. (c) For all $s \notin S_0(\sigma) \cup \{e\}$, $\sum_{j \in s \cap e} u_j - v_s > 0$. (d) There exists $\widetilde{\omega} > 0$ such that $(1 - \widetilde{q}_i)\, u_i = \widetilde{\omega}$ for all $i \in e$. ("Conditional equal loss" property.)*

The proof is in the appendix.

Note that, quite predictably, only the efficient coalition can succeed with probability one —point (a)—. There may be other coalitions that such deterministic *SCOOP*, σ, does not predict to form but have a worth equal to the sum of what their members obtain in the efficient coalition. That is, they belong to $S_0(\sigma)$. Then σ, predicts that each player obtains in that coalition what she obtains in the efficient coalition —point (b)—. This is a simple consequence of part (iii) of the definition of a *SCOOP*. The more interesting point (d) follows from the use of the Nash bargaining solution —together with consistency of beliefs— as a theory for surplus sharing inside each coalition. We will come back to this point below.

Non deterministic *SCOOP*s can be (partly) characterized in a similar way. Let us define

$$S_+(\sigma) = \{s \in S \mid p_s > 0\},$$

as the set of coalitions with positive probability, according to σ. Also, given σ, we define for each player i

$$q_i = \sum_{s \ni i} p_s,$$

the probability that player i belongs to the winning coalition. Then,

Lemma 4 *$\sigma = \{u_i^s, t_i^s, p_s, \lambda_s^r\}$ is a non-deterministic SCOOP only if it satisfies: (a) $u_i^s = u_i$ for all $s \in S_+(\sigma)$ and for all $i \in \cup S_+(\sigma)$. (b) $\sum_{j \in s} u_j - v_s = 0$ for all $s \in S_+(\sigma)$. (c) There exists $\omega > 0$ such that $(1 - q_i)\, u_i = \omega$, for all $i \in \cup S_+(\sigma)$. ("Equal loss" property.) (d) Let $u_k = \omega$ for any player $k \notin \cup S_+(\sigma)$, then $\sum_{j \in s} u_j - v_s \geq 0$ for all $s \notin S_+(\sigma)$.*

The proof is in the appendix.

Point (a) is again a direct consequence of part (iii) in the definition of a *SCOOP*, and part (b) is then an obvious consequence too. We now discuss with a little more detail the "equal loss property" in part (c), and the related "conditional equal loss property" in part (d) of the Lemma 3.

In a non deterministic *SCOOP* σ, the first line in part (i) of the definition of a *SCOOP* must define the players' payoffs for any coalition s in $S_+(\sigma)$. According to our theory (part (ii) of the definition of a *SCOOP*), the disagreement point for i in that coalition s, is $t_i^s = \frac{q_i - p_s}{1 - p_s} u_i$. Thus, the net surplus expected by each player in the coalition, $u_i - t_i^s$, is $(1 - q_i)\, u_i$ divided by $1 - p_s$. Since, following the *NBS*, this net surplus is equally divided among the members of s (and all coalitions have a player in common), this implies the equal loss property. Therefore, this important characteristic of non deterministic *SCOOP*s indeed follows from consistency of beliefs (part (ii)) and the *NBS* (part (i)).

In a deterministic *SCOOP*, we should rather pay attention to what is predicted conditional on the zero probability event that the efficient coalition e does not form. Conditional on that event, the *SCOOP* allows predicting positive probability for both, coalitions that can reach an agreement on how to share the surplus (i.e., in $S_0(\sigma)$) and coalitions that cannot (i.e., not in $S_0(\sigma)$). Nevertheless, coalitions not in $S_0(\sigma)$ result in zero payoff for the players involved, and so even conditioning on this event, the bargaining environment for players is analogous to the one in a non deterministic *SCOOP*. Thus, the conditional equal loss follows from the same principles as the equal loss property.

It is immediate that the point (c) in Lemma 3 —together with point (b)— implies that a deterministic *SCOOP* selects an imputation u in the core —when $e = N$—. A non deterministic *SCOOP* cannot be an imputation in the core, of course, but selects also a vector of payoffs that shares some properties with those of a deterministic *SCOOP* as we have remarked. Indeed, point (b) in Lemma 4 says that the payoff vector u is feasible in all coalitions that have positive probability, and cannot be challenged by any other coalition. That is, as in the case of a deterministic *SCOOP*, a non deterministic *SCOOP* selects a payoff vector that determines the sharing of the worth in all coalitions with positive probability, and is not *dominated*. The next subsection will consider this relationship of the *SCOOP* and the core, and the particular payoffs that the *SCOOP* selects in that set. As we will see, on this second count the predictions of the *SCOOP* are closely related to those of the Coalitional Nash Bargaining Solution (*CNBS*), proposed by Compte and Jehiel (2010). This relationship will also help showing that the *SCOOP* is (generically) unique and easy to compute.

3.2 Coalitional Nash Bargaining Solution (*CNBS*)

Compte and Jehiel (2010), CJ hereafter, define a concept, the *CNBS*, for games where $e = N$. For other games, it is only a slight abuse of their terminology to call *CNBS* the solution to Program P:

$$\begin{aligned} \max_{\{x_i\}_{i\in e}} \mathcal{N} &= \Pi_{i\in e} x_i \\ \text{s.t. } \textstyle\sum_{i\in e} x_i &\leq v_e, \qquad (1) \\ \textstyle\sum_{i\in s\cap e} x_i &\geq v_s, \ \forall s \neq e. \qquad (2) \end{aligned}$$

The solution exists and is unique, as the constraints define a convex set, and the objective function (the Nash product restricted to the efficient coalition) is a strictly quasi-concave function. When $e = N$, the choice set is the core of the game, and we recover the original definition in CJ. When $e \neq N$, the set of constraints, together with $x_i = 0\ \forall i \notin e$, defines the core of the *game with coalitional structure* $\mathcal{B}$, $(N, v, \mathcal{B})$, where $\mathcal{B} = \{e, \{i\}_{i\notin e}\}$. (See Aumann and Dreze, 1974.) In general, we will refer to this core of the game with coalitional structure when we talk of the core. As in CJ, the *CNBS* is then an imputation in the core that maximizes the Nash product for the members of e.[4]

Obviously, constraint (1) is binding. Also, let M be the set of constraints for which (2) is satisfied without slack, and let ϕ and δ_s be Lagrange multipliers associated with

[4] Also note that if e is smaller than N then the core of the game is empty but the core of the game with coalitional structure $\mathcal{B}$ may or may not be empty.

constraints (1) and (2) respectively. The solution satisfies

$$\frac{\mathcal{N}}{x_i} = \phi - \sum_{s \ni i} \delta_s, \quad \forall i \in e, \tag{3}$$

$$\sum_{i \in e} x_i = v_e, \tag{4}$$

$$\sum_{i \in s \cap e} x_i = v_s \ \forall s \in M, \tag{5}$$

and

$$\sum_{i \in s \cap e} x_i > v_s \ \forall s \notin M. \tag{6}$$

Thus, the system (3), (4), (5) has a solution (x, ϕ, δ_S) which is unique in x, but may not be unique in ϕ, δ_S. Generically, it is, but we need not exclude the cases where it is not. We will discuss some comparative statics of these multipliers, so we need to discuss this possible multiplicity. Let $m = |M|$. A sufficient condition for uniqueness of ϕ, δ_S is that the set of n_e–dimension vectors $A = (A_e, A_{s_1}, ..., A_{s_m})$ of partial derivatives of each of the constraints in M with respect to x_i for $i \in e$ are linearly independent: the linear independence constraint qualification would then be satisfied. Note that A_s, $s \in M$, is simply a vector with 1 in position i if player i belongs to s and 0 if she does not: what is called the *incidence vector* of s. For the (non-generic) cases where the qualification is not satisfied, Lemma 21 in the appendix offers sufficient information to obtain our results.

Following CJ, and starting from a game with a core that has a non-empty interior, consider the family of games that result from adding the same amount Δ, $\Delta \geq 0$, to the worths of all coalitions. Let $\mathcal{N}(\Delta)$ be the value of the Nash product (for players in e) at the *CNBS* for a given Δ. Now we can define the following property:

Property P1. $\frac{d\mathcal{N}(\Delta=0)}{d\Delta} = \phi - \sum_{s \neq e} \delta_s \geq 0.$

The nature of a *SCOOP* will depend on whether or not Property P1 holds.

As we will see, for any game with a non-empty interior of the core and whose *CNBS* does not satisfy Property P1, it is possible to define a modified game with a higher value of v_N that does satisfy Property P1 with equality —and with $e = N$—. This construct will be useful to prove uniqueness of the *SCOOP* in the next section. Thus, let us first define the modified game.

Definition 5 *$\widehat{v}(v)$ is a game where $\widehat{v}_s(v) = v_s$ for all $s \neq N$, and $\widehat{v}_N(v)$ is such that $\phi - \sum_{s \neq e} \delta_s = 0$ in Program P with $e = N$.*

The following lemma asserts that, whenever needed, $\widehat{v}(v)$ always exists and is unique.

Lemma 6 *For every game v that either has a core with an empty interior or a non-empty interior and a CNBS that violates Property P1,* $\widehat{v}(v)$ *exists and is unique.*

This lemma is proved in the appendix.

3.3 Computing the (generically) unique *SCOOP*

It will be useful to divide the space of games into two regions, which will correspond to games with deterministic and non deterministic *SCOOP*s, respectively. Region I comprises the games that have a core with a non-empty interior, and for which the *CNBS* satisfies Property P1. Region II comprises the games for which the core has an empty interior, or for which the *CNBS* satisfies $\phi - \sum_{s \neq e} \delta_s \leq 0$. Hence, these two regions exhaust the parameter space and only intersect in the frontier: games with a core whose interior is non-empty and whose *CNBS* satisfies $\phi - \sum_{s \neq e} \delta_s = 0$.

The next two propositions provide conditions on fundamentals that are necessary for the existence of deterministic and non deterministic *SCOOP*, respectively. They also provide the crucial information for obtaining these solutions.

Proposition 7 *A deterministic SCOOP exists only if the game belongs to Region I. Moreover, for a deterministic SCOOP,* $\sigma = \{u_i^s, t_i^s, p_s, \lambda_s^r\}$, $\{u_i^e\}$ *coincides with the CNBS and hence it is unique in payoffs and probabilities.*

Proof. Let $\sigma = \{u_i^s, t_i^s, p_s, \lambda_s^r\}$ be a *SCOOP* with $p_e = 1$. Then, it satisfies conditions (b) to (d) in Lemma 3. If we let $\mathcal{N} = \Pi_{i \in e} u_i$, then $x_i = u_i$, $\phi = \frac{\mathcal{N}}{\widetilde{\omega}}$, and $\delta_s = \lambda_s^e \frac{\mathcal{N}}{\widetilde{\omega}}$ for all $s \in S_0(\sigma)$, and $\delta_s = 0$ otherwise, satisfy conditions (3) to (5) with $M = S_0(\sigma)$. Since the objective function of Program P is strictly quasi-concave and the constraints are linear, these (first-order) necessary conditions for a maximum are also sufficient. Thus, $\{u_i\}$ in a deterministic *SCOOP* coincides with the *CNBS* —and so the core is not empty—. Finally, since $\sum_{s \in S_0} \lambda_s^e \leq 1$ and $\phi - \sum_{s \neq e} \delta_s = \frac{N}{\widetilde{\omega}} \left(1 - \sum_{s \neq e} \lambda_s^e\right) \geq 0$, Property P1 holds. This proves the first part of the proposition. As for the second part, since the *CNBS* is unique in payoffs, then we can only have a single deterministic *SCOOP* in payoffs and probabilities —since, by definition, $p_e = 1$—. ■

The arguments used in the proof also imply that conditions (a)-(d) in Lemma 3 are sufficient to identify payoffs and probabilities of the unique *SCOOP*. Note that conditions (a)-(d) rule out that the game is in Region II. (See Remark 11 below.)

Thus, a deterministic *SCOOP* predicts that players' payoffs coincide with the *CNBS*. That is, it predicts that the efficient coalition forms and the payoff vector is (the most egalitarian imputation) in the core, the one that maximizes the Nash product for players in that coalition.

A non-empty core is a necessary but not a sufficient condition for the existence of a deterministic *SCOOP*. Property P1 is necessary too. CJ identified this property as necessary for a random proponent-game to (asymptotically) have the *CNBS* as equilibrium payoffs. It will be useful to analyze the rational behind the necessity of Property P1 for negotiations to lead to a deterministic prediction.

If the efficient coalition is to form and distribute its worth among its members, it must be the case that, as the core requires, no subset of players can get a better deal forming some other coalition. That is, the worth of other coalitions should not be in the way of the proposed agreement. The *SCOOP* takes this a step further. A potential agreement in the efficient coalition may be destabilized not by (a deterministic prediction of the payoffs in) an alternative coalition —an outside option—, but by a counterfactual (conditional) probabilistic prediction of what coalition would form, and the corresponding payoffs in these potential coalitions. These predictions —beliefs—, however, must be consistent. As we have discussed, consistency requires in particular that all potential coalitions give players the same payoff as the efficient coalition. This immediately explains the relationship between (conditional) probabilities in the *SCOOP* and (relative) multipliers in the *CNBS*: i) since the *SCOOP* incorporates the *NBS* in each coalition, the payoff in the efficient coalition must result in maximizing the Nash product with appropriate constraints; ii) each coalition where the members may obtain the same payoff as in the grand coalition is a feasible outside option, and so "binds" the Nash product; i.e., will add to the disagreement point in the efficient coalition; and (iii) this disagreement payoff for each player is simply the payoff in the binding coalitions weighted with the coalitions' relative strengths: if we identify the conditional probability that each binding coalition forms, λ_s^e, with the relative weight of its Lagrange multiplier, δ_s/ϕ, the payoffs in a deterministic *SCOOP* must indeed be the *CNBS*. But consistency of beliefs requires that $\sum_{s\in M} \delta_s/\phi = \sum_{s\in M} \lambda_s^e \leq (\sum_{s\neq e} \lambda_s^e =)\ 1$, which is nothing but Property P1.[5]

[5] CJ make the connection between the multipliers in Program P and the —vanishing, as $\delta \to 1$— probability of success of each coalition in the equilibrium of the random-proponent protocol. Subgame perfection imposes equilibrium consistency on these probabilities. This most clearly illustrates how our consistency of counterfactuals is similar to the consistency of beliefs that is incorporated in subgame perfection. We will return to the consequences of this analogy in the Section 5.

We will further discuss this relationship, beginning with how this reasoning extends beyond Region I and so beyond *SCOOP*s that predict a deterministic outcome. Not surprisingly, Property P1 and the *CNBS* –this time, of modified games– will still play an important role.

Proposition 8 *A non-deterministic SCOOP exists only if the game belongs to Region II. Moreover, for a non-deterministic SCOOP,* $\sigma = \{u_i^s, t_i^s, p_s, \lambda_s^r\}$ *,* $\{u_i\}$ *is the CNBS for the modified game* $\widehat{v}(v)$*, and hence it is generically unique in payoffs and probabilities.*

Proof. Consider a game that lies in the interior of Region I; i.e., its generalized *CNBS* satisfies $\phi - \sum_{s \neq e} \delta_s > 0$. Suppose that $\sigma = \{u_i^s, t_i^s, p_s, \lambda_s^r\}$ is a non-deterministic *SCOOP* for this game, and so satisfies conditions (a) to (d) of Lemma 4. Hence, $\sum_{i \in e} u_i \geq v_e$. On the other hand, from part (iii) of the definition of a *SCOOP*, if $s \in S_+(\sigma)$ then $v_s = \sum_{i \in s} u_i \geq \sum_{i \in s \cap e} u_i$. Now, consider a modified game, $\widetilde{v}$, in which $\widetilde{v}_e = \sum_{i \in e} u_i$ and $\widetilde{v}_s = \sum_{i \in s \cap e} u_i$ for all $s \in S_+(\sigma)$, $s \neq e$, and $\widetilde{v}_s = v_s$ otherwise. Note that $x_i = u_i$ for all $i \in e$, $\phi = \frac{\mathcal{N}}{\omega}(1 - p_e)$, and $\delta_s = p_s \frac{\mathcal{N}}{\omega}$, for $\mathcal{N} = \Pi_{i \in e} u_i$, satisfy (3) to (6), with $M = S_+(\sigma) - \{e\}$, for game $\widetilde{v}$. Thus, u_i^e is the *CNBS* of $\widetilde{v}$. Therefore, for this game, $\sum_{s \neq e} \delta_s = \sum_{s \neq e} p_s \frac{\mathcal{N}}{\omega} = \phi$. Note that if v has a generalized core with a non-empty interior, then $\widetilde{v}$ does too, and also there exists a continuous path from v to $\widetilde{v}$. Thus, Lemma 21 applies. We obtained $\widetilde{v}$ from v by increasing v_e and reducing v_s. That is, by relaxing the resource constraint and tightening the rest of constraints, and so the *CNBS* of v must exhibit $\phi - \sum_{s \neq e} \delta_s \leq 0$. This is a contradiction which proves the first part of the Proposition.

Let us now prove generic uniqueness. Consider a game that lies in the interior of Region II, so that $\phi - \sum_{s \neq e} \delta_s < 0$. Suppose that we have two non deterministic *SCOOP*s, σ and σ'. Then, similar as before, consider the game $\widetilde{v}$, where $\widetilde{v}_N = \sum_i u_i$ and $\widetilde{v}_s = v_s$ for all $s \neq N$, and a similar game $\widetilde{v}'$ with $\widetilde{v}'_N = \sum_i u'_i$. Note that $x_i = u_i$, $\phi = \frac{\mathcal{N}}{\omega}$, and $\delta_s = p_s \frac{\mathcal{N}}{\omega}$, where $\mathcal{N} = \Pi_{i \in N} u_i$ satisfy (3) to (6) for $e = N$, with $M = S_+(\sigma) - \{N\}$,[6] and so $\{u_i\}$ is the *CNBS* of the game $\widetilde{v}$. Moreover, $\phi - \sum_{s \in M} \delta_s = 0$, and so $\widetilde{v} = \widehat{v}(v)$. The same could be stated for $\widetilde{v}'$, and since $\widehat{v}(v)$ is unique according to Lemma 6, we conclude that $\{u_i\} = \{u'_i\}$. Also, as discussed in Section 2.2. the set of coalitions M and the associated Lagrange multipliers are generically unique, which implies that almost always $p_s = p'_s$. In those non-generic cases that Lagrange multipliers are not unique, we still have that

[6] Recall that in non-generic cases, where M may not be uniquely defined, there may be multiple *SCOOP*'s, but all have the same payoffs in coalitions in $S_+(\sigma)$ and the same q_i.

individual probabilities of success are unique. In particular, condition (3) can be written as

$$\frac{\mathcal{N}}{u_i} = \phi(1 - \sum_{s \ni i} \frac{\delta_s}{\phi}) = \phi(1 - q_i),$$

for all $i \in N$. $\frac{\mathcal{N}}{u_i}$ are so is the marginal value of the resource constraint, ϕ, and so q_i are also unique. ■

Once more, the arguments used in the proof also imply that conditions (a)-(d) in Lemma 4 are sufficient to (generically) identify payoffs and probabilities of the unique *SCOOP*. Moreover, these conditions cannot be satisfied in Region I. (See Remark 12 below.)

When Property P1 is not satisfied or the interior of the core is empty, the prediction cannot be deterministic, but probabilistic. This implies that the "outside options" for players in one coalition with positive probability are the rest of coalitions with positive probability. Other than that, the logic is the same as in the case of a deterministic *SCOOP*. Thus, consider increasing the worth of the grand coalition so that it offers exactly the same payoff to each player as any of the positive probability coalitions in a non deterministic *SCOOP*. At that point, the grand coalition is —exactly— feasible given the *now* outside option represented by the original *SCOOP*. Moreover, the equal loss property guarantees that, with the disagreement point represented by the expected payoff in the original *SCOOP*, the (or rather, a) *SCOOP* of the new game is deterministic —the grand coalition is now the efficient one— and the payoffs for each player are the same as before, only with probability one. Moreover, λ_s^N in the new game coincides with p_s in the original one. That is, the *CNBS* of the new game coincides with the vector u in the original one and δ_s/ϕ in the new game coincides with p_s in the original one —and, of course, $\sum_{s \in S} p_s = 1$, since $\sum_{s \in M} \delta_s/\phi = 1$—.

By putting together Propositions 7 and 8, taking Theorem 2 into account, we conclude:

Theorem 9 *The SCOOP is generically unique. Moreover: (a) In the interior of Region I the unique SCOOP is deterministic, and (b) In the interior of Region II the unique SCOOP (in payoffs and individual probabilities of success) is non-deterministic.*

The frontier between Regions I and II (non-generic cases) requires special consideration. A game whose core has a non-empty interior and its *CNBS* satisfies $\phi - \sum_{s \neq e} \delta_s = 0$ exhibits multiple *SCOOP*. More specifically, there exists a single deterministic *SCOOP* and a continuum of non-deterministic *SCOOP*s. The deterministic *SCOOP* can easily be constructed from the *CNBS* following the same procedure discussed in the proof of Proposition 7 but in the opposite direction. That is, this *SCOOP* is defined by $p_e = 1$, $S_0(\sigma) = M$, $u_i^s = x_i$ for all $i \in e$ and $s \in M \cup \{e\}$, and $u_i^s = 0$ otherwise, $\lambda_s^e = \frac{\delta_s}{\phi}$, and $\lambda_s^r = 1$ if $r \neq e$.[7] Note that $\sum_{s\in S_0(\sigma)} \lambda_s^e = \sum_{s\in S_0(\sigma)} \frac{\delta_s}{\phi} = 1$. As for the non-deterministic *SCOOP*s, we can still use the *CNBS* (Note that $\widehat{v}_N(v) = v_e$) and construct a different *SCOOP* for each value of $p_e \in [0,1)$.[8]

In this non-generic case the multiplicity of *SCOOP*s is payoff-relevant. In particular, the probability of success of player i, that is, q_i, increases with p_e. In other words, all these *SCOOP*s are Pareto-ranked. The analysis of this frontier between the two regions also completes the characterization of the efficiency of the *SCOOP*:

Corollary 10 *An efficient SCOOP exists if and only if the game lies in Region I.*

This corollary is analogous to Propositions 1 and 2 in CJ. In particular, in the interior of Region I their random-proponent game (asymptotically) implements the *CNBS*. Thus, in this case the *SCOOP* confirms the *CNBS* as the right prediction. The reasons behind such coincidence are considered in the next section, where we further examine the nature of Property P1. Also, in Section 5 we discuss alternative non-cooperative protocols and their different predictions in Region II.

The above propositions also provide useful information for computing the *SCOOP*. In particular, if the game lies in Region I, the payoffs of the unique deterministic *SCOOP* can be computed by solving Program P. Equivalently, conditions (a) to (d) in Lemma 3 are also sufficient conditions to determine the payoffs (and probabilities) of the unique *SCOOP*. That is,

Remark 11 *If the triple* $(p_s, u_i^s, \lambda_s^e)$ *satisfies conditions (a)-(d) of Lemma 3, then* (p_s, u_i^e) *are the payoffs and probabilities of the unique deterministic SCOOP —where* t_r^s *and* λ_i^s *for* $s \neq e$ *are computed as in part (ii) in the definition of SCOOP—, and the game must lie in Region I.*

Computing the (probabilistic) SCOOP in Region II is almost as straightforward. Indeed, it involves solving Program P for a free parameter $\widehat{v}_N$, then solving for $\widehat{v}_N$ such that

[7] Disagreement points can be computed according to part II of the definition of *SCOOP*, and it is easy to check that such a quadruple is indeed a *SCOOP*.

[8] Indeed, consider $u_i^s = x_i$ for all $i \in e$ and $s \in M \cup \{e\}$ and $u_i^s = 0$ otherwise, any $p_e \in [0,1)$, and given p_e, $p_s = \frac{\delta_s}{\phi}(1 - p_e)$. Hence, $\sum_{s\in S} p_s = 1$. Since $p_s < 1$ for all s, then part (ii) of the definition of a *SCOOP* defines λ_r^s for all s and r. Given this, part (ii) of the definition of a *SCOOP* also defines t_i^s for all i and s. It is straightforward to check that we have indeed constructed non-deterministic *SCOOP*s.

$\phi - \sum_{s \neq e} \delta_s = 0$: the SCOOP puts positive probability δ_s/ϕ on coalitions for whihch this value is positive, and the corresponding payoffs u_i^s are equal to the solutions for x_i in this (modified) Program P. Equivalently, conditions (a) to (d) in Lemma 4 are sufficient to determine the payoffs and probabilities of the unique non-deterministic SCOOP. That is,

Remark 12 *If the pair (p_s, u_i^s) satisfies conditions (a)-(d) of Lemma 4, then they are the payoffs and probabilities of the generically-unique, non-deterministic SCOOP —where t_r^s and λ_i^s are computed as in part (ii) in the definition of SCOOP—. Also, the game must lie in Region II.*

4 Examples and discussion

In this section, we investigate the nature of predictions derived from the *SCOOP* as a solution concept. We begin by computing and discussing the $n = 3$ game. As we will show later, some of the characteristics of this case do not generalize to $n > 3$. Yet, besides being the simplest, this case has been important both in theoretical discussions —the three player-three cake game, (Binmore, 1985)— and also in applications —vertical contracting and foreclosure (Aghion and Bolton, 1987; Segal and Whinston, 2000), or mergers (Burguet and Caminal, 2015)—.

Without loss of generality, assume $v_{12} \geq v_{13} \geq v_{23}$. Also, let us begin with the strictly superadditive case, $v_{123} > v_{12}$. The interior of the core is non-empty if and only if $v_{123} > \frac{1}{2}(v_{12} + v_{13} + v_{23})$, and so only under this condition the *SCOOP* is deterministic, with the grand coalition succeeding with probability 1. We will illustrate how the conditions of Lemma 3 can be used to compute it, for each potential composition of the set S_0.

If $S_0 = \varnothing$ then $\widetilde{q}_i = 0$ for all i, so that condition (d) —the conditional equal loss— and condition (b) —the resource constraint— together imply that $u_i = \frac{v_{123}}{3}$. Condition (c) then shows that this is possible only if $v_{12} \leq \frac{2}{3} v_{123}$. Obviously, in this case conditional probabilities do not matter.

If $S_0 = \{(1, 2)\}$, then condition (b) implies that $u_1 + u_2 = v_{12}$, and condition (d) then implies $u_1 = u_2 = \frac{v_{12}}{2}$. Negotiations in the grand coalition imply that $u_3 = v_{123} - v_{12}$. Condition (c) requires that $u_1 + u_3 \geq v_{13}$; i.e., $v_{13} \leq v_{123} - \frac{v_{12}}{2}$. Finally, from condition (d), $\lambda_{12}^{123} = 3 - 2\frac{v_{123}}{v_{12}}$, which is always lower than 1, but positive only if $v_{12} \geq \frac{2}{3} v_{123}$.

The last case that we need to consider is $S_0 = \{(1, 2), (1, 3)\}$.[9] The system of equations

[9] If (12) is not an element of S_0, then $(1, 3)$ cannot be either: condition (d) would require that $u_3 \geq u_2$,

implied by condition (b) has one solution: $u_1 = v_{12} + v_{13} - v_{123}$, $u_2 = v_{123} - v_{13}$, and $u_3 = v_{123} - v_{12}$. (Condition (c) holds for coalition $(2,3)$, as $v_{123} > \frac{1}{2}(v_{12} + v_{13} + v_{23})$.) Finally, condition (d) implies

$$\begin{aligned}\lambda_{12}^{123} &= \frac{u_2(u_1 - u_3)}{u_2(u_1 - u_3) + u_1 u_3}, \\ \lambda_{13}^{123} &= \frac{u_3(u_1 - u_2)}{u_2(u_1 - u_3) + u_1 u_3} \leq \lambda_{12}^{123}.\end{aligned}$$

Thus, $\lambda_{12}^{123} + \lambda_{13}^{123} < 1$, and $\lambda_{13}^{123} > 0$ only when $v_{13} > v_{123} - \frac{v_{12}}{2}$.

Note that this exhausts the set of games that have a core with a non-empty interior. Also, it is immediate to check that we have in fact constructed the *SCOOP* for each case.

Next, we consider games with an empty core: $v_{123} < \frac{1}{2}(v_{12} + v_{13} + v_{23})$, and so with non-deterministic *SCOOP*. The conditions of Lemma 4 characterize the solution. Adding conditions (b) and (d) for all coalitions, we conclude that the grand coalition $(1,2,3)$ cannot belong to S_+. That is, $p_{123} = 0$. On the other hand, if any of the two player coalitions is not in S_+, then there is one player in all coalitions in S_+, and so $\omega = 0$, in condition (c). Condition (c) then implies that $u_i = 0$ for the other two players. But then condition (d) (i.e., $\sum_{j \in s} u_j - v_s \geq 0$) cannot hold for the efficient, grand coalition. Thus, $S_+ = \{(12), (13), (2,3)\}$, and $p_{ij} > 0$ for all i, j, and $\{u_i\}_{i=1,2,3}$ is the only solution to the linearly independent, three-equation, three-variable system of condition (b):

$$u_i = \frac{v_{ij} + v_{ik} - v_{jk}}{2}. \tag{7}$$

Note that $u_1 \geq u_2 \geq u_3 > 0$. The last inequality is a consequence of strict superadditivity. Also, condition (d) holds: $u_1 + u_2 + u_3 > v_{123}$. Condition (c) (the equal loss) allows to easily compute the probabilities:

$$u_i p_{jk} = u_j p_{ik} = u_k p_{jk} (= \omega). \tag{8}$$

Solving these two equations together with $p_{ij} + p_{ik} + p_{jk} = 1$, we obtain:[10]

$$p_{ij} = \frac{u_i u_j}{u_1 u_2 + u_1 u_3 + u_2 u_3}. \tag{9}$$

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and so $v_{12} \leq v_{13}$, which could only occur if $v_{12} = v_{13}$, a case in which the naming of these two coalitions would be arbitrary. Simlar arguments allow us to exclude the case that S_0 contains only $(2,3)$ and the case that it contains two coalitions other that $(1,2)$ and $(1,3)$. Finally, the three two-player coalitions can belong to S_0 only in the degenerate case that the four-equation linear system then implied by condition (b) has a solution, at which case $v_{123} = \frac{1}{2}(v_{12} + v_{13} + v_{23})$, and so the core has an empty interior.

[10]Since $u_1 \geq u_2 \geq u_3 > 0$ for all i, then $1 > p_{12} \geq p_{13} \geq p_{23} > 0$.

As discussed in the previous section, in the frontier of these two regions, $v_{123} = \frac{1}{2}(v_{12} + v_{13} + v_{23})$, there are multiple *SCOOP*s that can be Pareto-ranked, and indexed by $p_{123} \in [0,1]$. Payoffs are still given by equation (7) and probabilities of bilateral coalitions are equal to $(1 - p_{123})$ times the probabilities given by equation (9).

Let us now consider the non-superadditive case, $v_{12} > v_{123}$. The grand coalition cannot belong to neither S_0, if the *SCOOP* is deterministic, nor S_+, if it is non-deterministic. Indeed, in the former case, the grand coalition cannot satisfy condition (b) of Lemma 3 if coalition $e = (1,2)$ does, since the grand coalition contains all players in e. For the same reason, the grand coalition cannot satisfy condition (b) of Lemma 4 unless it is the only coalition in S_+ which, among other things, would contradict that the *SCOOP* is non-deterministic.

Thus, in the non-superadditive case, the *SCOOP* is almost trivial to compute, following the above procedure: (a) if $v_{12} < v_{13} + v_{23}$, then the *SCOOP* is non-deterministic and given by (7) and (9) above. If $2v_{13} \geq v_{12} \geq v_{13} + v_{23}$, then the *SCOOP* is deterministic, $S_0 = \{(1,3)\}$, and so $u_1 = v_{13}$ and $u_2 = v_{12} - v_{13}$, and otherwise, the *SCOOP* is deterministic and $u_1 = u_2 = \frac{v_{12}}{2}$.

Note that, in the non-superadditive case, $v_{12} > v_{13} + v_{23}$ is the condition for the core to have a non-empty interior. That is,

Remark 13 *When $n = 3$, an efficient SCOOP exists if and only if the core is not empty.*

This result means that, in the case $n = 3$, Property P1 is redundant: whenever the (interior of the) core is non-empty, Property P1 is satisfied. Indeed, if the interior of the core is non-empty, then there is at least one player who belongs to all coalitions in M. Thus, condition (3) for this player implies that the left hand side is strictly positive, and hence Property P1 holds.

For $n > 3$, even games that have a core with a non-empty interior may not have a deterministic *SCOOP*. The reason is that the core condition amounts to accepting that "all claims" can be realized with probability one —including claims that are mutually exclusive—. In contrast, the *SCOOP* requires the set of claims to be compatible. That is, that claims arise from a probability distribution over counterfactuals. The following (parametric) five-player example illustrates the failure of Property P1 and how such failure can be related to the inconsistency of counterfactuals:

Example 14 *Let $n = 5$, and suppose that only coalitions $e = (1,2,3)$, $s_1 = (1,4,5)$, and*

$s_2 = (2,4,5)$ *have positive worth:* $v_e = 1$, $v_{s_1} = v_{s_2} = \alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. *Note that the interior of the core is not empty.*[11] *Since* $\alpha > \frac{1}{3}$, *it is immediate that coalitions* s_1 *and* s_2 *must belong to* S_0 *in a deterministic SCOOP. This implies that* $u_1 = u_2 = \alpha$ *and* $u_3 = 1 - 2\alpha$. *The "conditional equal loss" property requires that*

$$u_1\left(1 - \lambda^e_{s_1}\right) = u_2\left(1 - \lambda^e_{s_2}\right) = u_3.$$

That is,

$$\lambda^e_{s_1} = \lambda^e_{s_2} = 3 - \frac{1}{\alpha}.$$

Therefore, since $\lambda^e_{s_1} + \lambda^e_{s_2} \leq 1$, *a deterministic SCOOP exists (if and) only if* $\alpha \leq \frac{2}{5}$. *That is, the SCOOP is not deterministic if* $\alpha > \frac{2}{5}$.

For an imputation to be in the core, which requires players 4 and 5 to obtain a payoff of 0, players 1 and 2 need be guaranteed at least α, what they can get in their own alternative coalition. This is the "claim" they may put forward when negotiating with player 3, and so player 3 has to accept a payoff of $1 - 2\alpha$.

However, these two claims are themselves in conflict: if player 1's claim was to be taken at face value, it would mean that, unless an agreement in the coalition e was reached, player 1 would manage to secure coalition s_1. Then player 2 would have no claim herself, since if coalition s_1 forms, then coalition s_2 cannot succeed too!

In the *SCOOP*, the claims of all players in a coalition must be consistent, and so the combined "claim" of players 1 and 2 in negotiations to form $(1,2,3)$ is at most α. The rest of the worth $v_e = 1$, i.e., at least $1 - \alpha$, has to be divided among the three players in e if this coalition is to succeed with probability 1. That is, player 3 must obtain at least $\frac{1-\alpha}{3}$, which is more than $1 - 2\alpha$ when $\alpha > \frac{2}{5}$. Thus, no core imputation can be the *SCOOP* in this case.

Also, if player 3 obtains at least $\frac{1-\alpha}{3}$, then players 1 and 2 combined cannot obtain more than $1 - \frac{1-\alpha}{3}$. Still, each must obtain at least α in coalition $(1,2,3)$ if this coalition is to form with probability 1. That is, for none of them to strictly prefer a feasible deal with players 4 and 5. Thus, $1 - \frac{1-\alpha}{3}$ must be larger than 2α, which is only possible if $\alpha \leq \frac{2}{5}$ (Property P1).[12]

[11] Property P1 can also fail in superadditive games with a non-empty core interior. CJ provide several examples. This example is simpler and thus can illustrate more transparently the role of consistent beliefs required by a *PSN*.

[12] If $\alpha > \frac{2}{5}$, the computation of the (non-deterministic) *SCOOP* is straightforward. Obviously, $u_1 = u_2$

Thus, the failure of Property P1 reflects the fact that sufficiently strong outside options may turn out to be non-credible in the negotiations of the efficient coalition. In order to examine this relation in more detail, consider again the solution to Program P. The payoff in the efficient coalition, the resource constraint (1), may be understood to satisfy whatever implicit claims players bring to that coalition. These claims can only come from the —binding— constraints in (2). Suppose the worth of all these coalitions increase by a —tiny— unit. If claims are consistent then the effect of that change on the total value of claims that players can bring to e cannot be larger than that unit. But then, the increase in the worth of the efficient coalition —the resource constraint— that is necessary for leaving the Nash product —the objective function— undiminished cannot be larger than this unit maximum increase in claims. And this is exactly the meaning of $\sum_{s \neq e} \delta_s / \phi \leq 1$ (Property P1).[13]

As an additional look at the role of consistency of claims in the *SCOOP*, it is of interest to consider how the relative payoff of each player depends on the worth of coalitions she is part of, particularly when the *SCOOP* is non-deterministic. That is, how the consistent claims of each player depend on the value of their potential alternatives. Looking at the computations for the case $n = 3$, it is immediate that if the value of a coalition increases, then none of its members obtains a lower expected payoff. However, this again is not general when $n > 3$. Indeed, consider the following example:

Example 15 *Let $n = 5$, and suppose that only five three-player coalitions can generate a positive surplus: $v_{123} = 1 + \Delta$, with $\Delta \in (0, 1)$, and $v_{145} = v_{245} = v_{134} = v_{235} = 1$. (Note that each player belongs to three relevant coalitions, and if Δ is very small then the value of these coalitions is very similar, and hence players are placed in a very symmetric position.) The core of the game is empty, so that the SCOOP is non-deterministic: S_+ includes all five coalitions, and using condition (b) in Lemma 4, $u_1 = u_2 = \frac{1+2\Delta}{3}$, and $u_3 = u_4 = u_5 = \frac{1-\Delta}{3}$. Using condition (c): $p_{123} = \frac{1+8\Delta}{5+4\Delta}$, $p_{145} = p_{245} = \frac{1+2\Delta}{5+4\Delta}$,and*

and $u_4 = u_5$. Moreover, if we let $p_e = p$, then $p_{s_1} = p_{s_2} = \frac{1-p}{2}$. Hence, using conditions (b) and (c) in Lemma 4, $u_1 = u_2 = \frac{2}{5}$, $u_3 = \frac{1}{5}$, $u_4 = u_5 = \frac{\alpha}{2} - \frac{1}{5}$, and $p = \frac{2}{5\alpha}$: the "excess value" of the outside claims $\left(\alpha - \frac{2}{5}\right)$ go to players 4 and 5, who can now obtain a positive payoff. That reduces the share of players 1 and 2 in these coalitions, which makes their claims in the efficient coalition consistent. As α increases players 1 and 2 obtain the same payoff conditional on success, but a lower probability of success. Since coalitions s_1 and s_2 become more relevant, their probability of success increase. And since $q_i = 1 - p_{s_j}$, $i, j = 1, 2$, $i \neq j$, a higher value of α implies a lower value of $q_1 = q_2$.

[13] In other words, this notion of consistent claims implies that a probability distribution can be constructed by assigning probabiity $\frac{\delta_s}{\phi}$ to coalitions that bind in (2) and imputing zero value to claims in any other coalition.

$p_{134} = p_{235} = \frac{1-4\Delta}{5+4\Delta}$, so that $q_1 = q_2 = 3\frac{1+2\Delta}{5+4\Delta}$, and $q_3 = q_4 = q_5 = \frac{3-4\Delta}{5+4\Delta}$. If Δ increases, then both $u_1 = u_2$ and $q_1 = q_2$ increase and hence players 1 and 2 are better off. However, u_3 and q_3 both decrease. Hence, a higher value of v_{123} benefits two of its members but hurts the third.[14]

Such —at first glance, perhaps counterintuitive— result reflects the key role of complementary coalitions in S_+. In the efficient coalition, (123), players 1 and 2 are the strong players, in the sense that they can also form their complementary coalitions, (145) and (245), respectively. That is, each of them can also form a coalition with the players absent in the efficient coalition, 4 and 5. In contrast, player 3 is a weak player in the efficient coalition since the value of her complementary coalition, (345), is zero. As v_{123} increases, the bargaining position of players 1 and 2 vis-a-vis players 4 and 5 in their respective coalitions with them, improves.[15] Thus, players 1 and 2 must gain —in terms of payoff conditional on the coalition succeeding— whatever players 4 and 5 loose in those coalitions, which is more than each player 4 and 5 alone loose. In contrast, player 3's alternative coalitions include one of her partners in the efficient coalition, the strong players $i = 1, 2$, together with only one weak player $j = 4, 5$. Thus, since player i gains more than player j loose, player 3 must also looses in those coalitions $(i3j)$.[16]

Player 3 not only looses in terms of payoff conditional on being in the succeeding coalition: the equal loss property in fact requires that $q_1 = q_2$ increase and $q_3 = q_4 = q_5$ decrease. That is, player 3 —and players 4 and 5— unambiguously looses as a result of an increase in the worth of a coalition she is part of!

5 A non-cooperative approach

What kind of non-cooperative bargaining protocol would implement the *SCOOP* as its unique equilibrium outcome? Small variations of the most popular protocols in the recent bargaining literature (in which agreements to form coalitions are irreversible) are not likely to do the job. In most of these protocols one player, who is chosen using an exogenous random device, proposes a particular coalition and how to share its value. Next, the

[14] If the value of any of the other coalitions increases then the comparative static results are analogous. For instance, if v_{235} increases then player 2 looses because she is the weak player ($p_{124} = 0$).

[15] In particular, players 1 and 2 must gain the same amount that is lost by players 4 and 5 together: $du_1 = du_2 = -(du_4 + du_5)$. Since the position of players 4 and 5 is symmetric, we have that $du_4 = du_5$ and hence $du_1 = du_2 = -2du_4 = -2du_5$.

[16] Since $du_1 + du_3 + du_4 = 0$ and $du_1 = -2du_4$, then $du_3 = du_4 = -\frac{1}{2}du_1$.

members of this coalition accept or reject the offer. Protocols vary depending on the consequences of rejections. In the *rejector-proposes* protocol (Chaterjee et al. 1993), the first player who rejects the offer becomes the next proponent. In the *random-proponent* protocol (CJ), if an offer is rejected the game moves to the next period, and the proponent is chosen again according to the random device used at the beginning. The specific initial random device affects the distribution of bargaining power beyond the fundamentals of the bargaining problem, and hence has a remarkable impact on individual payoffs and the probabilities of success of the different coalitions. It will be useful to illustrate this point by means of an example.

Consider a simple version of the three-player, three-cake problem where $v_{123} = 0$, $v_{12} = 1$, $v_{13} = v_{23} = \alpha \in (0, 1)$. If we use the random-proponent protocol with a uniform initial probability distribution, $\alpha \in (0.4, 0.7)$, and large discount factor, then there is an equilibrium in which the probability of success of the efficient coalition $(1, 2)$ is two thirds, precisely the probability that one of its members is selected as the proponent. Moreover, payoffs conditional on the success of any coalition are also independent of α (within that range). Thus, the random device introduces a great deal of rigidity: large changes in the fundamentals of the bargaining problem do not affect the prediction.

Moreover, in such an equilibrium there is a one third chance that an inefficient coalition succeeds. But when $\alpha \in (0.4, 0.5)$ the core is not empty and Property P1 holds, and so in this range there is also an efficient equilibrium. Thus, we conclude that this protocol is unable to make a unique prediction.

The *rejector-proposes* protocol introduces exogenous rigidities similar to the ones discussed above. [17]

We would argue that a sensible protocol should have predictions that are not hostage to exogenous, unjustified moves by Nature and instead respond to the primitives of the problem. One step in the right direction can be made by separating the determination of the winning coalition and the sharing of the value of a coalition. More specifically, consider the following bargaining game, that takes place over time: $t = 0, 1, 2, ...$(infinite horizon). All players discount the future using the same discount factor, δ. In period $t = 0$:

1. SELECTING THE COALITION

(1.1) Nature selects one player, i, using a probability distribution μ on N. Player i

[17] Also, it is well known (See, for instance, Proposition 3 in CJ), that such a protocol makes the implementation of the efficient coalition more difficult.

proposes a coalition, s.

(1.2) All other players involved in coalition s must accept or reject in sequence. a) If all players accept, the game moves to (2). b) The first player that rejects becomes the proponent in (1) in the next period.

2. SHARING THE VALUE OF THE COALITION

(2.1) Nature selects one of the members of coalition s, say j, as the proponent, using a uniform probability distribution. Player j makes a proposal to divide v_s.

(2.2) Players in s accept or reject in sequence. a) If all accept, then the proposal is implemented and the game ends. Otherwise, the game moves to the next period and starts fresh at (1.1).

If no agreement has been reached at the beginning of period t, then players continue bargaining using the protocol in $t = 0$, unless an offer was rejected in stage 1.2 of period $t-1$. In the latter case, the first player who rejected the offer is the current proponent in stage 1.1.

This describes a family of protocols, one for each probability distribution μ. Our results below show that there is one "protocol", that is, a distribution μ, that asymptotically — as $\delta \to 1$— implements the *SCOOP*. Moreover, any protocol μ for which a —subgame perfect, Markov— equilibrium exists does so. We begin with this latter result.

Proposition 16 *Suppose there exists* $\underline{\delta} \in (0,1)$ *such that the protocol* μ *has a subgame perfect, Markov equilibrium for* $\delta > \underline{\delta}$. *Then the outcomes of any selection of equilibria converge to the SCOOP.*

A formal proof of this result is in the appendix. Here, we offer a heuristic proof for the —perhaps less intuitive— case where the limiting equilibrium puts positive probability on more than one coalition.

Let δ be given, and denote the —equilibrium, continuation— payoff of player j by U_j. Whoever is the proponent in stage 2.1, say player i, will obtain a premium of $v_s - \sum_{j\in s} \delta U_j$, above her minimum payoff, δU_i, if that premium is positive. Thus, all players in $r \in \arg\max_s \omega_s \equiv \frac{1}{n_s}\left(v_s - \sum_{j\in s} \delta U_j\right)$ will be eager to accept this coalition if proposed in stage 1.1. Moreover, any player in any such coalition will reject in 1.2 any other coalition unless it is characterized by a ω_s sufficiently close to ω_r: they can always wait to the next period and propose coalition r. Since any other coalition will have at least one player in r, that means that all coalitions that are accepted with positive probability must be characterized by "sufficiently similar" values of ω_s. In fact, discounting is the only reason why any difference may be accepted. As $\delta \to 1$, this reason disappears, and so in the limit all coalitions that succeed with positive probability must be characterized by the same ω^*. That is, all of them must give the same payoff to player i, $x_i = U_i + \omega^*$.

On the other hand, if q_i is the equilibrium probability that player i is a member of the winning coalition, then as $\delta \to 1$, U_i approaches $q_i x_i$. Then, for any player who belongs to a coalition with a positive probability of success, $\omega^* = x_i - U_i = (1 - q_i)U_i$.[18] Then, for this case, and given Remark 12, the probabilities and payoffs must be those of the *SCOOP*, if the game lies in the interior of Region II.

Next, we show that there exist protocols with Markov equilibria, and so protocols that implement the *SCOOP*

Proposition 17 *There exist protocols μ such that for δ sufficiently large, a subgame perfect, Markov equilibrium exists. Thus, asymtotically, the equilibrium outcome is the SCOOP.*

The proof can be found in the appendix. It uses an approach in line with that used for the proof of Theorem 2 to prove, first, that there exist probability distributions over coalitions proposed and accepted in 1.1 and payoffs that are consistent with equilibrium for sufficiently large $\delta < 1$. Once these probability distributions are obtained, it is shown that there are mixing probabilities for each player over their proposals in 1.1, and "protocols", that is, Nature's probabilistic choice of who proposes in 1.1, μ, that result in those distributions over proposed and accepted coalitions. The approach underlines both the uniqueness of equilibrium outcomes and the continuum of protocols μ that, for a given cooperative game $\left(N, \{v_s\}_{s \in S}\right)$, asymptotically implement this unique outcome.

Thus, and returning to the discussion of rigidities, as long as a stationary equilibrium is possible, players' strategies in stage 1.1 will adjust to the protocol μ so that the equilibrium outcome will always coincide with our prediction, the *SCOOP*. That is, generically, small changes in the original protocol μ —the exogenous part— do not result in changes in the prediction, but in changes in behavior (proposals) so that the outcome remains unchanged, even when the prediction is probabilistic. On the contrary, small changes in

[18] If no coaltion forms with probability 1, then $\omega^* > 0$ unless we have an uninteresting case where some players are in all coalitions forming with probability 1 and the rest have all payoff zero. These are the cases that we exclude with the assumption of only one efficient coalition.

—relevant— primitives, like the worth of coalitions in S_+ or S_0 do change these predictions. This is in contrast with traditional protocols, as we have discussed above **and, we argue, something that should be considered a check of whether the protocol is sufficiently flexible to not restrict the outcome of the game**.

The protocol could be extended with the aim of **An alternative approach would be to attempt to** endogenizing μ. Given the above result, it is natural to expect a consensus among all players to emerge before stage 1.1. For instance, instead of assuming that Nature chooses a proponent in stage 1.1 using an exogenous random device, we could assume that, at the beginning of stage 1.1 all players simultaneously submit a probability distribution over proponents, μ_i. If there is unanimity, i.e., $\mu_i = \mu$ for all $i \in N$, then Nature uses such a random device to select the proponent. Otherwise, the game moves to the next period. Clearly, the unique stationary equilibrium of such an extended protocol asymptotically implements the *SCOOP*.

6 Concluding remarks

We have restricted attention to games in which all coalitions with a positive surplus involve more than one half of all players. When interested in studying games without externalities, it is natural to start focusing on this case since requiring that more than half of players are part of a relevant coalition trivially implies that only one coalition can be formed. Clearly, the problem of forming multiple coalitions is different and poses new questions. We can relax this restriction on coalition size, but still assume (following CJ) that only one coalition can be formed. That is, if a first coalition has formed, the value of all other coalitions becomes zero, which is an extreme specification of externalities. In this case, the *SCOOP* can still be a reasonable prediction. In particular, the *SCOOP* also exists for these games. In fact, Theorem 2 applies independently of any restriction on coalition sizes. However, we loose (generic) uniqueness. Such multiplicity can be illustrated using the following example.

Example 18 *$N = 5$, $v_N = 1$, $v_{12} = v_{34} = \alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, and $v_s = 0$ for any other s. A unique deterministic SCOOP exists if $\alpha < \frac{4}{9}$. However, if $\alpha > \frac{4}{9}$, there exists a continuum of non-deterministic SCOOP, one for each pair (p_{12}, p_{34}) satisfying $p_{12} + p_{34} = 1$, and so that p_{12} and p_{34} are higher than a minimum threshold Ω.[19] All these SCOOP share the*

[19] The value of this threshold is $\Omega(\alpha) = \frac{2(1-\alpha)}{5\alpha}$. Note that if $\alpha > \frac{4}{9}$ then $\Omega < \frac{1}{2}$.

same payoffs in the coalitions with positive probability of success: $u_1^{12} = u_2^{12} = u_3^{34} = u_4^{34} = \frac{\alpha}{2}$. *Also, part (iii) of the definition of a SCOOP implies that players* 1 *to* 4 *must expect a payoff in the grand coalition lower than* $\frac{\alpha}{2}$, *which is equivalent to* p_{12} *and* p_{34} *being higher than* Ω.

If coalitions with less than one half of players are sufficiently attractive, like in the previous example, then we can have two coalitions with positive probability of success that do not share any player. In this case, the *SCOOP* does not necessarily satisfy the equal loss property, which uniquely pinned down the probability distribution when $n_s > \frac{N}{2}$. Another way of putting it is that, in the absence of the equal loss property, the *SCOOP* is no longer the *CNBS* of a modified game that is uniquely defined.

7 References

Aghion, P., and P. Bolton (1987), Contracts as a Barrier to Entry. *American Economic Review*, 77(3), 388–401.

Aumman, R., and J. Dreze (1974), Cooperative GAmes with Coalition Structures. *International Journal of Game Theory*, 3(4), 217–237.

Bennett, E. (1997), Multilateral Bargaining Problems. *Games and Economic Behavior*, 19(2), 151–179.

Binmore, K. (1985), Bargaining and Coalitions. Chapter 13 in *Game-Theoretic Models of Bargaining*, Alvin Roth (editor). Cambridge: Cambridge University Press, 269-304.

Burguet, R., and R. Caminal (2015), Bargaining Failures and Merger Policy. *International Economic Review* 56(3), 1019–1041.

Chatterjee, K., B. Dutta, D. Ray, and K. Sengupta (1993), A Noncooperative Theory of Coalitional Bargaining. *Review of Economic Studies*, 60, 463–470.

Compte, O. and P. Jehiel (2010), The Coalitional Nash Bargaining Solution. *Econometrica*, 78(5), 1593–1623.

Hart, S. and A. Mas-Colell (1989), Potential Value and Consistency. *Econometrica* 57(3), 589–614.

Highham, J. (1990), Analysis of the Cholesky decomposition of a semi-definite matrix, in *Reliable Numerical Computation*, M. G. Cox and S. Hammarling, eds., Oxford University Press, pp. 161–185.

Peleg, B. (1986), "On the Reduced Game Property and its Converse", *International Journal of Game Theory* 15(3), 187–200.

Segal, I. R., and M. D. Whinston (2000), Exclusive Contracts and Protection of Investments. *RAND Journal of Economics*, 31(4), 603–633.

Sobolev, A. I. (1975), A characterization of optimality principles in cooperative games by functional equations. *Mathematical Methods in the Social Sciences* 6, 94–151 (in Russian).

8 Appendix

8.1 Proof of Theorem 2

Given a vector $\mathbf{p} = \{p_s\}_{s \in S}$, consider the equation

$$\mathbf{B} \cdot \mathbf{x} = -\mathbf{p} \cdot \mathbf{v} \cdot \mathbf{1}, \tag{10}$$

where $\mathbf{1}$ is the vector of dimension n with all components equal to 1, $\mathbf{v} = \{v_s\}_{s \in S}$ and $\mathbf{B}$ is the $n \times n$ matrix whose element $b_{ij} = -1$ if $i \neq j$ and

$$b_{ii} = n(1 - q_i) - 1,$$

where $q_i = \sum_{s \ni i} p_s$. Let Δ^ε denote the convex, compact set

$$\Delta^\varepsilon = \{\mathbf{p} \in \Delta : p_r \geq \varepsilon \ \forall r \in S\}.$$

where Δ is the simplex in $\mathbb{R}^{|S|}$. Also, let $\Delta^\cup = \cup_{\varepsilon > 0} \Delta^\varepsilon$, and let

$$\Delta^q = \{\mathbf{p} \in \Delta : q_i < 1 \ \forall i \in N\}.$$

Note that $\Delta^\cup \subset \Delta^q$. We first prove the following lemma.

Lemma 19 $\mathbf{B}$ *is invertible in* Δ^q*. Thus, (10) defines* $\mathbf{x}(\mathbf{p})$ *as an implicit, differentiable function in* Δ^q*.*

Proof. Let b_i denote the i^{th} diagonal element of $\mathbf{B}$. Substract the last row from all other rows of the matrix. The resulting matrix has zeros in all components of the rows 1 through $n-1$ except in the diagonal and in the last column. On row i, the diagonal element is $b_i + 1$, and the element in the last column $-(b_n + 1)$. Now multiply each row i, from 1 through $n-1$, by $\frac{1}{b_i+1}$ and add all of them to row n. We then have a triangular matrix (all components below the diagonal are zeroes). Thus, the eigen values of this matrix are the elements of the diagonal: $b_i + 1 = n(1 - q_i)$ for each $i < n$ and

$$b_n - (b_n + 1)\left[\sum_{i \neq n} \frac{1}{b_i + 1}\right]. \tag{11}$$

This eigen value is also non-zero. Indeed, we can write (11) as

$$\begin{aligned} & (b_n + 1)\left[\frac{\prod_{i=1}^n (b_i + 1) - \sum_{i=1}^n \prod_{j \neq i}(b_j + 1)}{\prod_{i=1}^n (b_i + 1)}\right] \\ = \quad & -(1 - q_n)\sum_{i=1}^n \frac{q_i}{1 - q_i} < 0 \end{aligned}$$

Since this matrix is obtained by row operations on $\mathbf{B}$, we conclude that $\mathbf{B}$ also has a non-zero determinant. ■

Note that $x_i(\mathbf{p})(1 - q_i)$ is the same for all players, and equals

$$w = \frac{\sum_j x_j(\mathbf{p}) - \mathbf{p} \cdot \mathbf{v}}{n}. \tag{12}$$

Also, we construct another function, $\mathbf{z} : \Delta^q \to \mathbb{R}^{|S|}$, based on $\mathbf{x}$, and defined as $z_s(\mathbf{p}) = \sum_{i \in s} x_i(\mathbf{p}) - v_s$. $\mathbf{z}(\mathbf{p})$ satisfies an important property: for any $\mathbf{p} \in \Delta^q$, $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$. Indeed, adding the n equations in (10),

$$\sum_{s \in S} p_s \left(\sum_{i \in s} x_i(\mathbf{p}) - v_s\right) = \sum_{i \in N} q_i x_i(\mathbf{p}) - \mathbf{p} \cdot \mathbf{v} = 0.$$

As a consequence, we cannot have that $z_s(\mathbf{p}) > 0$ for all $s \in S$. Also, by the same argument, we cannot have that $\mathbf{p}$ positive weight only on s such that $z_s(\mathbf{p}) < 0$.

To complete the definition of $\mathbf{x}(\mathbf{p})$ and $\mathbf{z}(\mathbf{p})$ for the case $\mathbf{p} \in \Delta - \Delta^q$, consider any sequence $\{\mathbf{p}(l)\} \to \mathbf{p}$, and define $\mathbf{x}(\mathbf{p}) = \lim_l \mathbf{x}(\mathbf{p}(l))$ and $\mathbf{z}(\mathbf{p}) = \lim_l \mathbf{z}(\mathbf{p}(l))$. Note that every point of Δ is a limit point of Δ^q and $\mathbf{x}$ is continuous, so this continuous extension is well defined. Also, note that $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ even for $\mathbf{p} \in \Delta - \Delta^q$.

Using these functions, and for arbitrary, given $\varepsilon > 0$, we construct a correspondence $h : \Delta^\varepsilon \twoheadrightarrow \Delta^\varepsilon$ as follows:

$$h_\varepsilon(\mathbf{p}) = \left\{\widetilde{\mathbf{p}} \in \Delta^\varepsilon : \widetilde{\mathbf{p}} \in \arg\min_{\overline{\mathbf{p}} \in \Delta^\varepsilon} \overline{\mathbf{p}}\mathbf{z}(\mathbf{p})\right\}.$$

Note that $\overline{\mathbf{p}}\mathbf{z}(\mathbf{p})$ is a linear function of $\overline{\mathbf{p}}$ and so $h_\varepsilon(\mathbf{p})$ is non empty and convex. Finally, $\mathbf{z}$ is continuous, and then trivially h_ε is upper hemi-continuous. Thus, from Kakutani's fixed point theorem, we conclude that h_ε has a fixed point in Δ^ε.

Some properties of the fixed points, and limit points of sequences of these fixed points are important here. Consider a sequence $\{\varepsilon(l)\} \to 0$ and a corresponding sequence of fixed points $\mathbf{p}(l)$ for the correspondence $h_{\varepsilon(l)}$. The sequence is in a compact set, Δ. Moreover, define $\widetilde{\lambda}_r^s(l) = \frac{p(l)_r - \varepsilon(l)}{1-p(l)_s} \in [0,1]$. $\widetilde{\boldsymbol{\lambda}}^s(l)$ is also a sequence in a compact set, $[0,1]^{|S|-1}$, for all s and l. Then, the sequence $\mathbf{p}(l)$ contains a subsequence that converges and where $\widetilde{\boldsymbol{\lambda}}^s(l)$ also converges for all s. Note that $\widetilde{\lambda}_r^s(l) \geq 0$ and

$$\lim_l \sum_{r \neq s} \widetilde{\lambda}_r^s(l) = \lim_l \frac{1 - p(l)_s - n_s \varepsilon(l)}{1 - p(l)_s} = 1$$

whenever $p_s < 1$. Thus, the limit $\mathbf{p}$ is a probability distribution, and the limits $\widetilde{\boldsymbol{\lambda}}^s$ when $p_s < 1$ are probability distributions and satisfy $\widetilde{\lambda}_r^s = \frac{p_r}{1-p_s}$. Moreover

Lemma 20 $\mathbf{z}(\mathbf{p}) \geq \mathbf{0}$.

Proof. Assume otherwise. That is, assume that $z_s(\mathbf{p}) = \beta < 0$. Continuity of $\mathbf{z}$ implies that for l large, $z_s(\mathbf{p}(l)) < \beta/2$. Thus, $p_r(l) = \varepsilon(l)$ for all r such that $z_r(\mathbf{p}(l)) > \beta/2$. Therefore, in the limit, $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) < 0$, which contradicts that, for all $\mathbf{p}$, $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$. ■

Given those limits $\mathbf{p}$ and $\widetilde{\boldsymbol{\lambda}}^s$, we now construct a candidate *SCOOP* as the four-tuple $\left\{\{u_i^s\}_{i \in s}, \{t_i^s\}_{i \in s}, p_s, \{\lambda_r^s\}_{r \neq s}\right\}_{s \in S}$ where:

1) p_s are the components of the vector $\mathbf{p}$;

2) $\boldsymbol{\lambda}^s = \widetilde{\boldsymbol{\lambda}}^s$ whenever $p_s < 1$. When $p_s = 1$, $\lambda_r^s = \widetilde{\lambda}_r^s$ if $z_r(\mathbf{p}) = 0$ and $\sum_{r \neq s} \lambda_r^s = 1$;

3) $u_i^s = x_i(\mathbf{p})$ if $z_s(\mathbf{p}) = 0$; if $z_s(\mathbf{p}) > 0$ then $u_i^s = 0$ if $v_s - \sum_{j \in s} q_j x_j(\mathbf{p}) < 0$ and $u_i^s = q_i x_i(\mathbf{p}) + \frac{1}{n_s}\left(v_s - \sum_{j \in s} q_j x_j(\mathbf{p})\right)$ if $v_s - \sum_{j \in s} q_j x_j(\mathbf{p}) \geq 0$;

4) $t_i^s = \sum_{r \ni i, r \neq s} \lambda_r^s u_i^r$, where u_i^r and λ_r^s are as defined in 2) and 3).

Note that in 2) we have not completely defined $\boldsymbol{\lambda}^s$ when $p_s = 1$. Indeed, any probability distribution $\boldsymbol{\lambda}^s$ satisfying what is required in 2) works, as we show below. It is important to notice that, given the definition of h_ε, $\widetilde{\lambda}_r^s(l) = 0$ for sufficiently large l, and so $\widetilde{\lambda}_r^s = 0$, whenever $z_r(\mathbf{p}) > 0$. Likewise, $\widetilde{\lambda}_r^s > 0$ only if $z_r(\mathbf{p}) = 0$, in which case $u_i^r = x_i(\mathbf{p})$. Also, even if $p_s = 1$, $\sum_{r \neq s} \widetilde{\lambda}_r^s \leq 1$.

We show that this four-tuple satisfies the definition of a *SCOOP*.

We begin by showing that (ii) holds, which is trivial: $t_i^s = \sum_{r \ni i, r \neq s} \lambda_r^s u_i^r$ for all $i \in s$, and $\lambda_r^s = \frac{p_r}{1-p_s}$ when $p_s < 1$.

We now show that (iii) holds. Note that $u_i^s = x_i(\mathbf{p})$ when $p_s > 0$, and so $z_s(\mathbf{p}) = 0$. According to 3), we only need to show that, when $z_s(\mathbf{p}) > 0$ (and so $p_s = 0$) and

$v_s - \sum_{j\in s} q_j x_j(\mathbf{p}) \geq 0$,

$$q_i x_i(\mathbf{p}) + \frac{1}{n_s}\left(v_s - \sum_{j\in s} q_j x_j(\mathbf{p})\right) \leq x_i(\mathbf{p}).$$

Note that

$$v_s - \sum_{j\in s} q_j x_j(\mathbf{p}) = v_s - \sum_{j\in s} x_j(\mathbf{p}) + \sum_{j\in s}(1-q_j)x_j(\mathbf{p}) \tag{13}$$

$$= \quad v_s - \sum_{j\in s} x_j(\mathbf{p}) + n_s(1-q_j)x_j(\mathbf{p}) = -\, z_s(\mathbf{p}) + n_s(1-q_j)x_j(\mathbf{p}),$$

(the second equality follows from the fact that $(1-q_j)x_j(\mathbf{p})$ is common for all players), and the result follows.

Finally, we prove that (i) holds. If $p_s < 1$, then $\lambda_r^s > 0$ only when $z_r(\mathbf{p}) = 0$, and so when $u_r^i = x_i(\mathbf{p})$, so that

$$\sum_{j\in s} t_j^s = \sum_{j\in s} x_i(\mathbf{p}) \sum_{r\ni i, r\neq s} \lambda_r^s = \sum_{j\in s} x_i(\mathbf{p}) \frac{q_i - p_s}{1-p_s} = \sum_{j\in s} x_i(\mathbf{p})(1 - \frac{1-q_i}{1-p_s}).$$

Thus, if $v_s < \sum_{j\in s} t_j^s$, then $v_s < \sum_{j\in s} x_i(\mathbf{p})$, i.e., $z_s(\mathbf{p}) > 0$, and so $u_i^s = 0$. Otherwise, i.e., if $v_s \geq \sum_{j\in s} t_j^s$, but still $z_s(\mathbf{p}) = 0$, we have

$$u_i^s - t_i^s = x_i(\mathbf{p}) \frac{1-q_i}{1-p_s} = u_j^s - t_j^s,$$

for all $i, j \in s$, where the last equality follows from the fact that $x_i(\mathbf{p})\,(1-q_i)$ is common to all players. This plus $z_s(\mathbf{p}) = 0$ implies again that (i) is satisfied. On the other hand, if $v_s \geq \sum_{j\in s} t_j^s$, and $z_s(\mathbf{p}) > 0$ (so that $p_s = 0$) then $u_i^s - t_i^s = x_i(\mathbf{p})(1-q_i)$, and also

$$v_s - \sum_{j\in s} t_j^s = v_s - \sum_{j\in s} x_i(\mathbf{p})(1-q_i).$$

Thus, since again $x_i(\mathbf{p})\,(1-q_i)$ is common to all players, (i) is satisfied.

The only case left is $p_s = 1$ (and so $z_s(\mathbf{p}) = 0$). As before, we show that $u_i^s - t_i^s = u_j^s - t_j^s$ for all $i, j \in s$. Indeed, note that

$$x_i(\mathbf{p}(l))\,(1-q_i(l)) = x_i(\mathbf{p}(l))\left(1 - \sum_{r\ni i} p_r(l)\right)$$

$$= \quad x_i(\mathbf{p}(l))\,(1-p(l)_s)\left(1 - \sum_{r\ni i, r\neq s} \frac{p(l)_r - \varepsilon(l)}{1-p(l)_s}\right)$$

$$= \quad (1-p(l)_s)\, x_i(\mathbf{p}(l))\left(1 - \sum_{r\ni i, r\neq s} \widetilde{\lambda}_r^s(l)\right).$$

Thus, since $x_i(\mathbf{p}(l))\,(1-q_i(l))$ is common to all players, $x_i(\mathbf{p}(l))\left(1-\sum_{r\ni i,r\neq s}\widetilde{\lambda}^s_r(l)\right)$ is common for all players in s, and so this is also the case in the limit. But $z_s(\mathbf{p})=0$, so that $u^s_i=x_i(\mathbf{p})$, and also, $t^s_i=\sum_{r\ni i,r\neq s}\lambda^s_r u^r_i=\sum_{r\ni i,r\neq s}\widetilde{\lambda}^s_r(l)x_i(\mathbf{p})$, since $u^r_i=0$ whenever $\widetilde{\lambda}^s_r=0$ and $\lambda^s_r>0$, and $u^r_i=x_i(\mathbf{p})$ whenever $\widetilde{\lambda}^s_r=\lambda^s_r>0$, as we noted before. Therefore, indeed $u^s_i-t^s_i=u^s_j-t^s_j$ for all $i,j\in s$.

Thus, it only remains to be shown that, in the limit,

$$v_s\geq\sum_{j\in s}t^s_j=\sum_{j\in s}x_i(\mathbf{p})\sum_{r\ni j,r\neq s}\widetilde{\lambda}^s_r,$$

which is immediate since $\sum_{r\neq s}\lambda^s_r\leq 1$.

8.2 Proof of Lemma 3

Let $p_r=1$. Thus, $t^s_i=u^r_i$ for all $i\in r\cap s$, $s\neq r$, and $t^s_i=0$ otherwise. Now, $\sum_{i\in r}t^r_i\leq v_r$: otherwise $u^r_i=0$, so that $t^s_i=0$ for all s, and so that $u^s_i>0$ for s such that $v_s>0$, which contradicts part (iii) of the definition of *SCOOP*. Hence, $\sum_{i\in r}u^r_i=v_r$, from (i) in de definition of a *SCOOP*. Let $u_i=u^r_i$ for all $i\in r$. If $\sum_{i\in s}t^s_i$ $(=\sum_{i\in s\cap r}u_i)<v_s$ then for all $i\in s\cap r$, $u^s_i>u_i$, which contradicts part (iii) of the definition of a *SCOOP*. Hence, $\sum_{j\in s\cap r}u_j-v_s\geq 0$ for all s. Also, if $s\neq r$ and $\sum_{i\in s}t^s_i$ $(=\sum_{i\in s\cap r}u_i)=v_s$, then from part (i) of the definition of a *SCOOP*, $u^s_i=u_i$ if $i\in s\cap r$ and $(t^s_j=)\ u^s_j=0$ if $j\notin r$. Finally, also from part (i) of the definition of a *SCOOP*, $u^s_j=0$ for all j if $\sum_{i\in s}t^s_i>v_s$, and this concludes the proof of (b), if $r=e$. From the definition of $S_0(\sigma)$, and since $\sum_{j\in s\cap r}u_j-v_s\geq 0$ for all s, we have that $\sum_{i\in s}t^s_i=\sum_{i\in s\cap r}u^r_i>v_s$, which proves (c).

Suppose $r\neq e$. Then,

$$v_r=\sum_{i\in r}u^r_i\geq\sum_{i\in r\cap e}u_i\geq v_e,$$

and reach a contradiction, since e is the unique efficient coalition. This proves (a) and completes (b).

Note that, from (b)[20], $t^e_i=\widetilde{q}_i u_i$. Hence,

$$u_i=\widetilde{q}_i u_i+\frac{1}{n_e}\left(v_e-\sum_{j\in e}\widetilde{q}_j u_j\right).$$

Letting $\widetilde{\omega}=\frac{1}{n_e}\left(v_e-\sum_{j\in e}\widetilde{q}_j u_j\right)\geq 0$, we have $(1-\widetilde{q}_i)\,u_i=\widetilde{\omega}$ for all $i\in e$. If $\widetilde{\omega}=0$ then $\widetilde{q}_i=1$ for all $i\in e$ such that $u_i>0$. Thus, if $S_0(\sigma)\neq\varnothing$, for any coalition $s\in S_0(\sigma)$ we

[20] In the non-generic cases where $e\in S_0$, we should interpret the (undefined) λ^e_e as 0.

have $\sum_{i\in e} u_i = \sum_{i\in s\cap e} u_i$ and so

$$v_e = \sum_{i\in e} u_i = \sum_{i\in s\cap e} u_i = v_s,$$

which contradicts that there is only one efficient coalition. On the other hand, if $S_0(\sigma) = \varnothing$, then $t_i^e = 0$ for all $i \in e$, and so $\widetilde{\omega} > 0$, unless $u_i = 0$ for all $i \in e$. Since $v_e > 0$, this cannot be, and this proves (d).

8.3 Proof of Lemma 4

In a random *SCOOP*, $|S_+(\sigma)| \geq 2$. From part (iii) in the definition of *SCOOP*, any player that is in more than one coalition in $S_+(\sigma)$ must be indifferent between them. Hence, $u_i^s = u_i$ for all $i \in s$ and all $s \in S_+(\sigma)$. This is point (a). Suppose that $v_s - \sum_{i\in s} t_i < 0$ for all $s \in S_+(\sigma)$. Then, $u_i^s = 0$ for all $i \in s$ and $s \in S_+(\sigma)$. This implies that all $t_i^r = 0$, for all r, and so $u_i^s > 0$ for s such that $v_s > 0$, from part (i) of the definition of *SCOOP*, which is a contradiction. Now suppose that $v_s - \sum_{i\in s} t_i < 0$ for some $s \in S_+(\sigma)$, so that $u_i^s = 0$ for all $i \in s$. From part (a), this implies that $u_i = 0$ for all players in s, and so $\sum_{i\in s} t_i = 0$ which contradicts $v_s - \sum_{i\in s} t_i < 0$. Thus, $\sum_{j\in s} u_j - v_s = 0$, which is point (b).

Consider a coalition $s \in S_+(\sigma)$. Since $t_i^s = \frac{q_i - p_s}{1-p_s} u_i$ for all $i \in s$, then from (i) in the definition of a *SCOOP* we can write:

$$u_i = \frac{q_i - p_s}{1-p_s} u_i + \frac{1}{n_s}\left(v_s - \sum_{j\in s} \frac{q_j - p_s}{1-p_s} u_j\right),$$

which, using $\sum_{j\in s} u_j = v_s$ from (b), is equivalent to

$$(1-q_i)\, u_i = \frac{1}{n_s}\left(v_s - \sum_{j\in s} q_j u_j\right).$$

The right hand side is common to all players $i \in s$ and since any two coalitions in $S_+(\sigma)$ share at least one player, then $(1-q_i)\, u_i = \omega$ for all players $i \in \cup S_+(\sigma)$, for $\omega = \frac{1}{n_s}\left(v_s - \sum_{i\in s} q_i u_i\right) \geq 0$. We will show below that in fact $\omega > 0$, and so will complete the proof of (c).

Consider now a coalition $s \notin S_+(\sigma)$. If $v_s - \sum_{i\in s} t_i^s \geq 0$, then for all $i \in s$,

$$u_i^s = q_i u_i + \frac{1}{n_s}\left(v_s - \sum_{j\in s} q_j u_j\right).$$

Equivalently,

$$(1-q_i)\,u_i^s = \frac{1}{n_s}\left(v_s - \sum_{j\in s} q_j u_j\right) \equiv \omega_s. \tag{14}$$

From part (iii) of the definition of *SCOOP*, $u_i^s \le u_i$ for all $i \in \cup S_+(\sigma)$. Hence,

$$(1-q_i)\,u_i \ge \omega_s. \tag{15}$$

Also, let $u_k = \omega$ obtained above for every player $k \notin \cup S_+(\sigma)$, i.e., so that $q_k = 0$. Thus, $u_i \ge u_i^s$ for all i, and then, adding (15) for all $i \in s$, and taking (14) into account,

$$\sum_{i\in s}(1-q_i)\,u_i \ge n_s\omega_s = v_s - \sum_{i\in s} q_i u_i,$$

or, equivalently,

$$\sum_{i\in s} u_i \ge v_s.$$

This is point (d). Finally, we show that $\omega > 0$. Indeed, suppose $\omega = 0$. This implies that $q_i = 1$ for all $i \in \cup S_+(\sigma)$ such that $u_i > 0$. That is, $u_i > 0$ only for players in $\cup S_+(\sigma)$ that belong to all coalitions in $S_+(\sigma)$. Since $S_+(\sigma)$ contains at least two coalitions (i.e., at least one r coalition other than e), and from (b),

$$v_r = \sum_{i\in r} u_i \ge \sum_{i\in e} u_i \ge v_e.$$

This contradiction proves that $\omega > 0$ and completes the proof of point (c).

8.4 Lemma 21

The following lemma is used in the proos:

Lemma 21 *If the core of the game has a non-empty interior, then* $\phi - \sum_{s\neq e}\delta_s$ *is unique and differentiable.*

The lemma is proved in three steps

Step 1 Program P is equivalent to a —different— optimization program with the same objective function and equality constraints with linearly independent Jacobian, A.

Proof. Let the Jacobian of the system of active constraints in one solution to Program P be A. Obviously, the solution x^* to Program P is also the solution to an alternative program with only constraint (1) and equality constraints corresponding to M. Suppose that A is not a linearly independent system, so that A_s, for some $s \in M \cup \{e\}$ can be written as

$$A_s = \sum_{s' \in M \cup \{e\} - s} \alpha_{s'} A_{s'},$$

for some vector $\alpha = \{\alpha_{s'}\}_{s' \in M \cup \{e\} - s}$. That means that

$$x \cdot A_s = x \cdot \sum_{s' \in M \cup \{e\} - s} \alpha_{s'} A_{s'},$$

for any vector of payoffs x. In particular, it is satisfied by the solution x^*, so that

$$v_s = x^* \cdot A_s = x^* \cdot \sum_{s' \in M \cup \{e\} - s} \alpha_{s'} A_{s'} = \sum_{s' \in M \cup \{e\} - s} \alpha_{s'} v_{s'}$$

Therefore, any vector x that satisfies the constraints corresponding to coalitions $s' \neq s$, i.e, such that $x \cdot A_{s'} = v_{s'}$ also satisfies

$$x \cdot \sum_{s' \in M \cup \{e\} - s} \alpha_{s'} A_{s'} = \sum_{s' \in M \cup \{e\} - s} \alpha_{s'} x A_{s'} = \sum_{s' \in M \cup \{e\} - s} \alpha_{s'} v_{s'} = v_s,$$

and so satisfies the constraint related to s. Thus, the constraint corresponding to coalition s is redundant. We can therefore exclude this coalition from $M \cup \{e\}$, and the new program would still have the same solution x^*. Repeating this argument, if needed, we conclude that there exists a set $M' \cup \{e\}$ of constraints such that the problem defined by (1) and (2) with only these —equality— constraints has x^* as a solution and its Jacobian is a linearly independent system. ■

Step 2 Let (x^*, ϕ, δ) and (x^*, ϕ', δ') be two solutions to system (3), (4), (5). Then, $\phi - \sum_{s \in S - \{e\}} \delta_s = \phi' - \sum_{s \in S - \{e\}} \delta'_s$. Moreover, if Property P1 is satisfied, then both ϕ and $\sum_{s \neq e} \delta_s$ are unique.

Proof. (3) can be written as

$$A \cdot \begin{bmatrix} \phi \\ -\delta \end{bmatrix} = \frac{\overrightarrow{\mathcal{N}}}{x^*}, \tag{16}$$

where, similarly as before, A is the $n_e \times (|M| + 1)$ matrix with component $a_{is} = 1$ if $i \in s$ and 0 otherwise, for $s \in M \cup \{e\}$, δ is the $|M|$ dimension vector with components δ_s, and $\frac{\overrightarrow{\mathcal{N}}}{x^*}$ is the n_e dimension vector with components all equal to $\frac{\mathcal{N}}{x_i^*}$. If A has rank $|M| + 1$ ($\leq n_e$), then the solution, which exists, is unique. Then Step 2 is trivially satisfied. Otherwise, one of the columns in A is a linear combination of the rest. Consider two possible cases. First, if A_e is not spanned by the rest of columns of A. In this case, it is not spanned by any subset of them, of course, and if the rank of A is smaller than $|M|+1$ it must be because some the set of $|M|$ columns other than A_e is not a linearly independent system. That is, one of these other columns of A, call it A_r,can be obtained as a linear combination of the columns in A other than A_r and A_e. Then, system (16) can be written as

$$\begin{aligned} a_{ie}\phi - \sum_{s\neq r} a_{is}\delta_s - \delta_r\left(\sum_{s\neq r} a_{is}\alpha_s\right) &= \frac{\mathcal{N}}{x_i^*} \Longleftrightarrow \\ a_{ie}\phi - \sum_{s\neq r} a_{is}\left(\delta_s + \alpha_s\delta_r\right) &= \frac{\mathcal{N}}{x_i^*}, \end{aligned}$$

for some vector $\alpha \in \mathbb{R}^{|M|-1}$. Thus, ϕ and $\widehat{\delta}_s = \delta_s + \alpha_s\delta_r$, together with x^* solves the system (3), (4), (5), when in (5) we consider the constraints in M except the one corresponding to r. Note that

$$\phi - \sum_{s\in M-\{s\}} \widehat{\delta}_s = \phi - \sum_{s\in M} \delta_s.$$

This, process may be repeated until the system M is reduced to a linearly independent subsystem. Thus, (3), (4), (5) with constraints only in this subsystem are satisfied by the solution –in x^* and multipliers– to (1) and (2) with this subsystem. Since that subsystem has a unique solution, and the solution to (1) and (2) with this set of constraints exists, we conclude that this unique solution is $x^*, \phi, \widehat{\delta}$, and the lemma follows.

Now suppose that A_e is spanned by the rest of columns of A. As before, system (16) can be written as

$$\begin{aligned} \phi\left(\sum_{s\in M} a_{is}\alpha_s\right) - \sum_{s\in M} a_{is}\delta_s &= \frac{\mathcal{N}}{x_i^*} \Longleftrightarrow \\ \sum_{s\in M} a_{is}\left(-\delta_s + \alpha_s\phi\right) &= \frac{\mathcal{N}}{x_i^*}, \end{aligned}$$

for some vector $\alpha \in \mathbb{R}^{|M|}$, so that $\widehat{\phi} = 0$ and $\widehat{\delta}_s = \delta_s - \alpha_s\phi$, together with x^* is a new solution to system (3), (4), (5), when in (5) we consider only the constraints in M except the one corresponding to A. Note that, again

$$-\sum_{s\in M} \widehat{\delta}_s = \phi - \sum_{s\in M} \delta_s. \tag{17}$$

Repeating the procedure until we obtain a set of linearly independent constraints, and recalling that still a solution to (1) and (2) must exist, and be characterized as the unique solution to (3), (4), (5) when considering the surviving constraints, we obtain that again $\phi - \sum_{s\in M} \delta_s$ is unique. Then, Property P1 contradicts (17), since $\widehat{\delta}_s \geq 0$ for all $s \in M$. That finishes the proof of Step 2. ■

Step 3 $\phi - \sum_{s\neq e} \delta_s$ is a differentiable function of v_e and v_s.

Proof. Consider the problem in (1) and (2) with only —equality— constraints in (a linearly independent set of constraints) M'. Let x be a solution to this problem. That is, there exists a positive real number ϕ and a vector δ in $\mathbb{R}_+^{m'}$, where $m' = |M'|$ such that (ϕ, δ, x) maximizes

$$L(\phi, \delta, x) = \Pi_i x_i - \phi(\textstyle\sum_{i\in e} x_i - v_e) + \sum_{s\in M'} \delta_s \left(\sum_{i\in s} x_i - v_s\right).$$

The first order conditions for this problem are

$$\begin{aligned} G_i &\equiv \frac{\partial \mathcal{N}}{\partial x_i} - \phi + \textstyle\sum_{s\ni i} \delta_s = 0, \\ G_s &\equiv \textstyle\sum_{i\in s} x_i - v_s = 0, \\ G_e &\equiv -\textstyle\sum_{i\in e} x_i + v_e = 0, \end{aligned}$$

where, once more, $\mathcal{N} \equiv \Pi_i x_i$. Totally differentiating with respect to (ϕ, δ, x) and v_e, we obtain the system

$$D \cdot \begin{bmatrix} dx \\ d\delta \\ d\phi \end{bmatrix} = \begin{bmatrix} -\frac{\partial G}{\partial v_e} dv_e \end{bmatrix}, \tag{18}$$

where D is the Hessian matrix of $\mathcal{N}$, H, bordered with the Jacobian of the constraints, A:

$$D = \begin{bmatrix} H & A \\ A' & 0 \end{bmatrix};$$

dx is the vector of size n_e with components dx_i; $d\delta$ is the vector of dimension m' with components $d\delta_s$, and $\frac{\partial G}{\partial v_e}$ is the $n_e + m' + 1$ dimension vector with terms $\frac{\partial G_i}{\partial v_e}$, $\frac{\partial G_s}{\partial v_e}$, and $\frac{\partial G_e}{\partial v_e}$. Note that H is invertible. Indeed, the ij entry in H is $\frac{\mathcal{N}}{x_i x_j}$ if $i \neq j$ and 0 if $i = j$. Thus, multiplying each row i by $\frac{x_i}{\mathcal{N}}$ and each column j by x_i, all non zero values, we have a matrix with entries 1 for all ij with $i \neq j$ and 0 if $i = j$. The determinant of this matrix is $(-1)^{n_e - 1}(n_e - 1) \neq 0$, and so the determinant of H is also non zero. Thus,

$$\det D = \det D/H \det H,$$

where D/H is the Schur complement of H, $-A'H^{-1}A$. Note that D/H is full rank. Indeed, $-H^{-1}$ is positive semidefinite and invertible, and so there exists a permutation matrix P such that

$$-P'H^{-1}P = R'R$$

for some upper triangular matrix R with all elements in the diagonal strictly positive (see, for instance, Highham 1990). Thus,

$$-A'H^{-1}A = -A'PP'H^{-1}PP'A = -A'PR'RP'A.$$

Note that $A'P$ is simply a permutation of A', and so has rank m, and then $A'PR'$ has also rank m. Thus, the rank of the product of that matrix and its transpose is m, and we conclude that $\det D/H > 0$ and $\det D \neq 0$. Thus, (ϕ, δ, x) is indeed a differentiable function of v_e and of v_s. ■

8.5 Proof of Lemma 6

Consider any given game v with a non-empty core so that Property 1 does not hold: $\phi - \sum_{s \neq e} \delta_s < 0$. Consider the game v' defined by $v'_s = v_s$ for all $s \neq N$, and $v'_N = n \times v_e + \epsilon$. The *CNBS* for this game is $u_i = \frac{v'_N}{n}$ $\forall i$ (the *NBS*), $\phi > 0$, and $\delta_s = 0$ for all $s \neq N$. Thus, $\phi - \sum_{s \neq e} \delta_s > 0$. Since the multipliers are continuous in v_N from Lemma 21, and $\phi - \sum_{s \neq e} \delta_s$ is monotne in v_e, the result follows. Now suppose the core of v is empty. Still, the game v'_N has a non empty core. Moreover, there exists a minimum value $v''_N < v'_N$ for which the core is not empty: the core is monotone in v_N. Thus, the result follows unless the game v'', defined with that value v''_N, satisfies Property P1 with strict inequality. We can rule out this possibility: The Nash product cannot increase if we increase —at least— one binding coalition payoff by as much as what we increase the payoff in v_N, when the interior of the core is empty.

8.6 Proof of Proposition 16

For simplicity, suppose that players that are indifferent between accepting and rejecting accept in (2.2). Suppose that we have an equilibrium of the game, and let Z_+ be the set of coalitions that have positive probability of forming in equilibrium. That means that all players —may— accept in 1.2) if a coalition $s \in Z_+$ is proposed. Also, let φ_s be the probability that s is formed in equilibrium, and x_i^s the expected payoff of player i if coalition s is proposed in 1.1) and everybody accepts in 1.2. Finally, let U_i be the equilibrium payoff; that is, the expected payoff of payer i at 1) before Nature selects: $U_i = \sum_{s \ni i} \varphi_s x_i^s$. Note that, for any $s \in Z_+$, that is, any s that succeeds with positive probability,

$$\begin{aligned} x_i^s &= \frac{1}{n_s}\left(v_s - \delta \sum_{j \neq i, j \in s} U_j + \delta(n_s - 1) U_i\right) \\ &= \delta U_i + \frac{1}{n_s}\left(v_s - \delta \sum_{,j \in s} U_j\right), \end{aligned} \tag{19}$$

and so $x_i^s - \delta U_i = x_j^s - \delta U_j$ for any $i, j \in s$. Define $w_s = \frac{1}{n_s}\left(v_s - \delta \sum_{,j\in s} U_j\right)$ this common value, and let $s^* = \arg\max_{s \in Z_+} w_s$. The first necessary condition for equilibrium is that for all $s \in Z_+$

$$\delta w_s \leq w_{s'} \leq w_s / \delta. \tag{20}$$

To show this, first $w_{s'} \geq \delta w_{s^*}$ since otherwise, when s' is proposed in 1.1, a player $j \in s' \cap s^*$ profits from rejecting and proposing in the next period: $x_i^{s^*} = w_{s^*} + \delta U_i$ is maximal for all $i \in s^*$ and so every player in s^* will accept in equilibrium. Note that such j that belongs to both s^* and s' always exists. Second, $\delta w_{s^*} \geq \delta w_s$ from the definition of s^*. Thus, we conclude that $\delta w_s \leq w_{s'}$. Switching the roles of s and s', we obtain that $\delta w_{s'} \leq w_s$, and (20) follows. Note that our discussion also implies that $s \notin Z_+$ if $w_{s'} < \delta w_{s^*}$; that is, if

$$v_s < \delta \sum_{j \in s} \left(w_{s^*} + U_j\right). \tag{21}$$

Now, (19) can be written as

$$x_i^s = \delta \varphi_s x_i^s + \delta \sum_{z \neq s, z \ni i} \varphi_z x_i^z + \frac{1}{n_s}\left(v_s - \delta \varphi_s \sum_{j \in s} x_j^s - \delta \sum_{j \in s} \sum_{z \neq s, z \ni j} \varphi_z x_j^z\right),$$

which, taking into account that $\sum_{j \in s} x_j^s = v_s$, substracting in both sides $\delta \varphi_s x_i^s$ and then dividing both sides by $(1 - \delta \varphi_s)$, can be written as

$$x_i^s = \delta \sum_{z \neq s, z \ni i} \frac{\varphi_z}{1 - \delta \varphi_s} x_i^z + \frac{1}{n_s}\left(v_s - \delta \sum_{j \in s} \sum_{z \neq s, z \ni j} \frac{\varphi_z}{1 - \delta \varphi_s} x_j^z\right). \tag{22}$$

With these conditions, suppose that our protocol has an equilibrium for each δ, and consider any selection of such equilibria. Also, consider any sequence $\delta_n \to 1$ and a limit point of the corresponding sequence of equilibrium probabilities φ_s, φ_s^∞. Note that $\sum_{s \in S} \varphi_s^\infty = 1$. Consider the set Z_+^∞ of coalitions such that $\varphi_s^\infty > 0$. From (20), and since $x_i^s - \delta U_i = w_s$, we have

$$x_i^s \to x_i \ \forall s \in Z_+^\infty,$$

for some values x_i. Thus, w_s for all $s \in Z_+^\infty$ converge as well. Let w^∞ be this limit of convergence. Finally, from (22), and if $\varphi_s^\infty < 1$ for all s,

$$x_i = \sum_{z \neq s, z \ni i} \frac{\varphi_z^\infty}{1 - \varphi_s^\infty} x_i + \frac{1}{n_s} \left(v_s - \sum_{j \in s} \sum_{z \neq s, z \ni j} \frac{\varphi_z^\infty}{1 - \varphi_s^\infty} x_j \right). \tag{23}$$

Now, suppose that indeed $\varphi_s^\infty < 1$ for all s. From (21), $\varphi_s^\infty = 0$ if $v_s < \sum_{i \in s} x_i$ if we let $x_i = w^\infty$ for all $i \notin \cup Z_+^\infty$. Moreover, adding (23) for all $i \in s$ for $s \in Z_+^\infty$, we have $v_s = \sum_{i \in s} x_i$. Also, from the definition of U_i, $x_i^s - \delta U_i \to x_i(1 - \sum_{s \ni i} \varphi_s^\infty)$ and $w_s \to w^\infty$ for all i (and all s). Thus, if we let $u_i^s = x_i$ for all i and $s \in Z_+^\infty \equiv S_+(\sigma)$, $p_s = \varphi_s^\infty$, for all s, and $\omega = w^\infty$, the conditions (a)-(d) of Lemma 4 are satisfied. Then, from Remark 12, we are in Region II and (x_i, φ_s^∞) are the payoffs and probabilities of the *SCOOP*.

Contrary, suppose $\varphi_s^\infty = 1$ for some s. Thus, $\delta U_j \to 0$ for all players not in s. Therefore, $s = e$ as otherwise, for δ sufficiently large one player in e and s would gain by rejecting in 1.2 and then proposing e in the following period. Also, for n large, $e \in Z_+$, and $x_i^e - \delta U_i = x_i^e - \delta x_i^e = w_e$. Thus, $w_e \to 0$, and so, from (20), $w^\infty = 0$. Consider the sequence $\left\{ \frac{\varphi_s}{1 - \delta \varphi_e} \right\}_{s \neq e}$ and note that $\frac{\varphi_s}{1 - \delta \varphi_e} < 1$. Thus, the sequence is bounded. Consider also any limit point of this set (sequence), $\{\lambda_s^\infty\}_{s \neq e}$, and any subsequence of δ_n for which the corresponding $\left\{ \frac{\varphi_s}{1 - \delta \varphi_e} \right\}_{s \neq e}$ converges to it. In that subsequence, $v_s - \delta_n \sum_{,j \in s} U_j \leq w_{s^*}$, and so in the limit $v_s - \sum_{,j \in s} x_j \leq w^\infty = 0$ for all s. On the other hand, if $\lambda_s^\infty > 0$, then $\varphi_s > 0$ for n large. That is, $s \in Z_+$ and so $w_s \geq 0$, i.e., $v_s - \delta_n \sum_{,j \in s} U_j \geq 0$, and we conclude that $v_s - \delta_n \sum_{,j \in s} U_j = 0$. Finally, (23) for $s = e$ implies that $(1 - \widetilde{q}_i)\, x_i$ is common to all players in e, where $\widetilde{q}_i = \sum_{s \ni i, s \neq e} \lambda_s^\infty$. Thus, conditions (a)-(d) of Lemma 3 are satisfied with only letting $\widetilde{\omega} > 0$ be that common value and $S_0(\sigma)$ the set of coalitions for which $\lambda_s^\infty > 0$. Thus, remark 11 implies that we are in Region I and also that the payoffs and probabilities are those of the *SCOOP*.

8.7 Proof of Proposition 17

We will restrict to equilibria where $x_i^s = u_i$ for all $s \in Z_+ = S_+$, where u_i is as defined in the *SCOOP*. Note that (22) implies a modified version of the equal loss property. Indeed, for any two players in s, the last term is common, and so equilibrium requires that

$$u_i(1 - \delta \sum_{s \ni i, s \in S_+} \varphi_s) = w$$

for all $i \in \cup S_+$. Thus, we begin by proving that this system has a solution $\{\varphi_s\}_{s\in S_+}$ with $\sum_{s\in S_+} \varphi_s = 1$. That is, that a positive solution exists for the linear system

$$C_+ \cdot \begin{bmatrix} w \\ \varphi \end{bmatrix} = \begin{bmatrix} u \\ 1 \end{bmatrix}, \tag{24}$$

where u is the N-dimension vector with components u_i, and φ is the m-dimension vector with components φ_s, with $m = |S_+|$, and C_+ is the $(N+1)\times(m+1)$ matrix

$$C_+ = \begin{bmatrix} 1 & \delta I(1,s_1)u_1 & ... & \delta I(1,s_m)u_1 \\ ... & ... & ... & ... \\ 1 & \delta I(N,s_1)u_N & ... & \delta I(N,s_m)u_N \\ 0 & 1 & .. & 1 \end{bmatrix},$$

where $I(i,s)$ is an indicator function that takes the value of 1 if $i \in s$ and 0 otherwise. Note that, for $\delta = 1$, this system has a solution: the probabilities p_s and the "loss" w in the *SCOOP*. Let this solution be (w^*, φ^*) Also note that the space spanned by the rows of the matrix C_+ is the same as the one spanned by the matrix $\widehat{C}_+$ obtained by multiplying each of its N rows by $1/\delta$. $(u,1)$ belongs to this space: in this basis $\widehat{C}_+$, $(u,1)$ has coordinates $(\delta w^*, \varphi^*)$. Thus, $(u,1)$ belongs to the space spanned by C_+ and so a solution to the system (24) indeed exists. Moreover, since $\varphi_s^* > 0$, the new coordinates (w,φ) are all also positive for sufficiently large δ.

Let θ_s^i denote the probability that player i chooses coalition s if chosen by Nature in step 1.1. We now show that there exists a protocol μ and probabilities θ_s^i such that, for all i and δ sufficiently large: proposing according to θ_s^i in 1.1; accepting in 1.2 if $s \in S_+$; rejecting otherwise and then proposing next period according to θ_s^i; accepting any offer —weakly— above $\delta \sum_{s\ni i} \varphi_s u_i$ in 2.2, and rejecting otherwise; offering that much if in 2.1, and $s \in S_+$ (and otherwise asking for v_s); is an equilibrium and

$$\varphi_s = \textstyle\sum_{i\in N} \mu_i \theta_s^i. \tag{25}$$

If μ_i and θ_s^i exist such that (25) is satisfied, and since $x_i^s = u_i$ for all $s \in S_+$, the result would follow as long as $\theta_s^i = 0$ if $i \notin s$. The existence of such solutions is trivial. For instance, given φ, we may set

$$\begin{aligned} \mu_i &= \textstyle\sum_{s\ni i, s\in S_+} \frac{\varphi_s}{n_s}, \\ \theta_s^i &= \begin{cases} \frac{\varphi_s}{n_s}\frac{1}{\mu_i} & \text{if } i \in s \in S_+ \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Note that $\mu_i = 0$ if $i \notin \cup S_+$. Also, if $i \in \cup S_+$, then $\sum_{s \in S} \theta_s^i = 1$, and also

$$\sum_{i \in N} \mu_i = \sum_{i \in N} \sum_{\substack{s \ni i \\ s \in S_+}} \frac{\varphi_s}{n_s} = \sum_{s \in S_+} \varphi_s = 1.$$

Summarizing, if μ is such that there exists a stationary equilibrium of the non-cooperative game, then in the limit such protocol implements the *SCOOP*.