Robust Predictions in Dynamic Policy Games*

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Abstract

Dynamic policy games feature a wide range of equilibria. This paper provides a methodology for obtaining robust predictions. We begin by focusing on a model of sovereign debt although our methodology applies to other settings, such as models of monetary policy or capital taxation. The main result of the paper is a characterization of outcomes that are consistent with a subgame perfect equilibrium conditional on the observed history. Our methodology provides observable implications across all equilibria that we illustrate by characterizing, conditional on an observed history, the set of all possible continuation prices of debt and comparative statistics for this set; by computing bounds on the maximum probability of a crisis; and by obtaining bounds on means and variances. In addition, we propose a general dynamic policy game and show how our main result can be extended to this general environment.

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1 Introduction

Following Kydland and Prescott (1977) and Calvo (1978), the literature on optimal government policy without commitment has formalized interactions between a large player (government) and a fringe of small players (households, lenders), dynamic policy games, by building on the tools developed in repeated games in the work of Abreu (1988) and Abreu et al. (1990). This agenda has studied interesting applications for capital taxation (e.g., Chari and Kehoe, 1990, Phelan and Stacchetti, 2001, Farhi et al., 2012), monetary policy (e.g., Ireland, 1997, Chang, 1998a, Sleet, 2001) and sovereign debt (e.g., Calvo, 1988, Eaton and Gersovitz, 1981, Chari and Kehoe, 1993, Cole and Kehoe, 2000) and helped us to understand the distortions introduced by lack of commitment and the extent to which governments can rely on reputation to achieve better outcomes.

One of the challenges in studying dynamic policy games is that these settings typically feature a wide range of equilibria with different predictions over outcomes. For example, there are "good" equilibria where the government may achieve, or come close to achieving, the optimum with commitment, while there are "bad" equilibria where this is far from the case, and the government may be playing the repeated static best response. When studying dynamic policy games, which of these equilibria should we employ? Can we make any general prediction given this pervasive equilibrium multiplicity? One approach is imposing refinements, such as various renegotiation-proof notions, that either select an equilibrium or significantly reduce the set of equilibria. Unfortunately, no general consensus has emerged on the appropriate refinements.

The goal of this paper is to overcome the challenge multiplicity raises by providing predictions in dynamic policy games that hold across all equilibria; following the terminology of Bergemann and Morris (2013), *robust predictions*. The approach we offer involves making predictions for future play that depend on past play. The key idea is that even when little can be said about the *unconditional* path of play, quite a bit can be said once we *condition* on past observations. To the best of our knowledge, this simple idea has not been exploited as a way of deriving robust implications from the theory. Formally, we introduce and study a concept which we term "equilibrium consistent outcomes": outcomes of the game, after an observed history, that are consistent with all subgame perfect equilibria that on its path could have generated the observed history.

Although the notions we propose and results we derive are general and apply to a large class of dynamic policy games, for concreteness we first develop them for a specific application, using a model of sovereign debt along the lines of Eaton and Gersovitz (1981). In the model, a small open economy faces a stochastic stream of income. To smooth consumption, a benevolent government can borrow from international debt markets, but lacks commitment to repay. If it defaults on its debt, the only punishment is permanent exclusion from financial markets; it can never borrow again. There are two features of this model that make it appealing to our work. First, this model has been widely adopted and is a workhorse in international economics. Second, as we show in this paper, this policy game can feature wide equilibrium multiplicity. On one end of the spectrum, in the worst equilibrium, the government is in autarky, facing a price of zero for debt issuance, and consuming its income. Meanwhile, in the best equilibrium, the government smooths consumption, and there is no room for self-fulfilling crises.[1]

Our main result provides a characterization of equilibrium consistent outcomes in any period (debt prices, debt issuance, and default decisions). Aided by this characterization, we obtain bounds for equilibrium consistent debt prices that are history dependent. The highest equilibrium consistent price is the one of the best equilibrium, is Markovian and, thus, independent of past play. The lowest equilibrium consistent price is strictly positive and depends on past play. Due to the recursive nature of equilibria, only the previous period play matters and acts as a sufficient statistic for the set of equilibrium consistent prices. The fact that the last period is a sufficient statistic may seem surprising. However, this result is a direct expression of robustness: it can always be the case the expected payoff rationalizing a decision had to be realized in histories that have not occurred.

The restrictions that we obtain in this paper are intuitive. In our sovereign debt application, equilibrium consistent debt prices improve whenever the government avoids default under duress. In particular, if the country just repaid a high amount of debt, or did so under harsh economic conditions, for example, when output was low, the lowest equilibrium consistent price is higher. The choice to repay under these conditions reveals an optimistic outlook for bond prices that narrows down the set of possible equilibria for the continuation game. This optimistic outlook is the expression of a *dynamic revealed preference* argument. What the government has left on the table as a consequence of its past

[1] Given that our approach tries to overcome the challenges of multiplicity, we first ensure that there is multiplicity in the first place. In particular we show that in the standard Eaton and Gersovitz (1981) model, restrictions on debt, which are often adopted in the quantitative sovereign-debt literature (Chatterjee and Eyigungor 2012 and micro-founded in Amador, 2013), can imply the existence of multiple equilibria (see Auclert and Rognlie, 2016 for necessary and sufficient conditions for uniqueness). Our multiplicity relies on the existence of autarky a subgame perfect equilibrium. This result may be of independent interest, since it implies that rollover crises are possible in this setting. The quantitative literature on sovereign debt following Eaton and Gersovitz (1981) features defaults on the equilibrium path, that are caused by shocks to fundamentals. A recent exception is Stangebye (2018) that studies numerically the role of nonfundamental shocks in sovereign crises using a model as in Eaton and Gersovitz (1981) with long term debt. Another strand of the literature studies self-fulfilling debt crises following the models in Calvo (1988) and Cole and Kehoe (2000). Our results suggest that crises, defined as episodes where the interest rates are very high but not due to fundamentals, may be a robust feature in models of sovereign debt.

decisions, reveals the expectations over the future. In equilibrium, these expectations over the future must be correct, and hence imposes restrictions over future outcomes, which are the basis of the predictions we obtain in this paper.

What is the importance of obtaining robust predictions? What we describe as robust predictions in this paper, which follows the terminology in Bergemann and Morris (2013), can also be described as the observable implications of equilibrium. In two influential papers, Jovanovic (1989) and Pakes et al. (2015), characterize for static games the observable implications of models with multiple equilibria. These implications, which are based on a *static revealed preference* argument, have been the basis of large literature in Industrial Organization and Econometrics that utilize them to estimate models with multiple equilibria (see Tamer, 2010 and De Paula, 2013 for recent reviews). To the best of our knowledge, ours is the first paper to obtain predictions over observables in a dynamic model with multiple equilibria without appealing to any equilibrium selection. We believe that our main results could be used as the basis of estimation techniques for dynamic models without imposing assumptions regarding the class of equilibria.

The first part of the paper characterizes equilibrium outcomes for the model as in Eaton and Gersovitz (1981). One of the limitations of this analysis is that in the classic version of the model, there is a deterministic relationship between the goverment's policies and prices. There are many reasons to think that this link is not that tight. In fact, a large literature in sovereign lending, but also a large body of work studying other dynamic policy games, has focused on the implications of breaking this link (at least since Calvo, 1988 and Cole and Kehoe 2000). Thus, we study a variation of the model that allows for coordination failures and crisis, by introducing a sunspot variable that is realized after the government chooses its policies but before market prices are realized.

For this generalized version of the model our main result, following the classic approach in Aumann (1987a), characterizes probability distributions over outcomes, what we term as "equilibrium consistent distributions". Even though in the model enriched with sunspots any *equilibrium* price can now be realized after a particular equilibrium history, we show that there are bounds on the probability distributions over prices. This is intuitive. For example, if the government just repaid a large amount of debt, it cannot be consistent with an equilibrium that they receive a price of zero with probability one. This intuition will be the basis of the characterization of equilibrium consistent distributions. This characterization is based on the same dynamic revealed preference argument that we explained above, which is a consequence of sequential rationality and that beliefs are correct in equilibrium.

As in the baseline model, building on the characterization of equilibrium consistent distributions we then turn to explore the predictions on observables that hold across all equilibria. First, we obtain bounds on the maximum probability of low prices; for example, a rollover debt crises, a price of zero. Due to equilibrium multiplicity, as we argued above, rollover debt crises may occur on the equilibrium path for any fundamentals. However, the probability of a rollover crisis, after a certain history, may be constrained. We derive these constraints, showing that rollover crises are less likely if the borrower has recently made sacrifices to repay. Second, we study bounds on moments of distributions over outcomes. In particular, we characterize bounds over the expected value of debt prices given a history for any equilibrium. Surprisingly, the bounds of expected prices of debt will be tightly related to the bound of prices in the model without sunspots, which are easy to compute. In addition, as in Bergemann et al. (2015), we characterize bounds on variances, which holds across all equilibria. As we mentioned before, the importance of bounding moments across all equilibria is that these can be the basis of econometric estimation methods.

In the last section of the paper we show how our characterization of equilibrium consistent outcomes extends to a more general class of dynamic policy games. In particular, we provide a general model of credible government policies, which follows the seminal contribution of Stokey (1991). The key features that the general setup tries to capture are lack of commitment, a time inconsistency problem, infinite horizon that creates reputation concerns in the sense of trigger-strategy equilibria, and short run players that form expectations regarding the policies of the government. With some variation on the timing of the moves for the players, most dynamic policy games share these features. After proposing the general model, and showing that widely used frameworks such as the model of Eaton and Gersovitz (1981) and the New Keynesian model as in Woodford (2011), fit in the setup, we replicate our main results of the paper for this general setup.

Literature Review. Our paper relates to several strands of the literature. First, to the literature on credible government policies. The seminal papers on optimal policy without commitment are Kydland and Prescott (1977) and Calvo (1978). Applications range from capital taxation as in Phelan and Stacchetti (2001) and Farhi et al. (2012); monetary policy as in Ireland (1997), Chang (1998a), Sleet (2001) and Waki et al. (2015); and sovereign debt Atkeson (1991), Arellano (2008), Aguiar and Gopinath (2006), Cole and Kehoe (2000), and more recently Dovis et al. (2017). We believe that our paper is closely related to Chari and Kehoe (1990), Stokey (1991) and Atkeson (1991). The first two papers adapt the techniques developed in Abreu (1988) to characterize completely the set of equilibria in dynamic policy games. Atkeson (1991) extends the techniques in Abreu et al.

(1990), by allowing for a stochastic public state variable, in the context of sovereign lending finding interesting properties of the best equilibrium. Our paper studies a related, yet different question. Instead of characterizing equilibria at the beginning of the game, we characterize continuation equilibria given a history of play. This characterization of continuation equilibria is precisely the basis for obtaining predictions that are robust across all equilibria. Our central assumption is that *an* equilibrium has generated the history of play, without appealing to any equilibrium refinement.

Second, to the literature on robust predictions. The papers that are more closely related to our work are Angeletos and Pavan (2013), Bergemann and Morris (2013) and Bergemann et al. (2015). The first paper, Angeletos and Pavan (2013), obtains predictions that hold across every equilibrium in a global game with an endogenous information structure. The second paper, Bergemann and Morris (2013), obtains restrictions over moments of observable endogenous variables that hold across every possible information structure in a class of coordination games. In a related paper, Bergemann et al. (2015) characterize bounds on output volatility, across all potential information structures, in a static model where agents face both idiosyncratic and common shocks to productivity. Our paper contributes to this literature by obtaining predictions that hold across all equilibria in a dynamic game. In particular, we obtain restrictions over the distribution of equilibrium debt prices, for any possible process of sunspots, by exploiting the dynamic implications that sequential rationality has on the distribution of observables. These implications are the basis to obtain bounds on first and second order conditional moments, across all possible sunspot processes, or following the terminology in Bergemann and Morris (2017), across all possible *information structures*.[2] These bounds provide testable implications of the model, even in the presence of both equilibrium multiplicity and uncertainty of the information structure agents have when making their decisions.

Third, our paper relates to the literature that studies the observable implications of models with multiple equilibria. The two more close related papers are Jovanovic (1989) and Pakes et al. (2015). The first paper, Jovanovic (1989), provides a framework to discuss conditions under which a model with multiple equilibria is point or set identified. The main ideas are clearly illustrated in a two person entry game, one of the canonical examples of estimation of games with multiple equilibria.[3] The second paper, Pakes et al.

[2]The literature of information design in dynamic games, where agents may have access to private information about other players actions, was first formalized by Myerson (1986) and Forges (1986), extending the concept of correlated equilibrium to extensive form games. Extending the intuition of Aumann (1985), Forges (1986), and most recently Sugaya and Wolitzky (2017) in an incomplete information setting.

[3]Entry games have been studied extensively in the IO literature (see for example Bresnahan and Reiss, 1990, Berry, 1992, Bajari et al., 2007, Ciliberto and Tamer, 2009), or are examples of a large literature on estimation of static and dynamic games of complete (see for example Aguirregabiria and Mira, 2007 and

(2015), discuses conditions under which inequality constraints can be used as a basis for estimation and inference.[4] Both papers are based on a revealed preference argument that places bounds over observables given an optimizing behavior of an agent. Our paper, is based on a dynamic version of this revealed preference argument: what the government just left on the table, reveals an outlook for the future, and this outlook for the future places bounds over observables. The importance of obtaining dynamic observable implications is that extends the applicability of the previous results, which focus on a static setting.

Finally, sections 2, 3, and 4 of this paper study robust predictions in a dynamic policy game that builds on Eaton and Gersovitz (1981). This framework, and variations of it, have been extensively used to study sovereign borrowing. The literature has followed two main directions. One direction, the quantitative literature on sovereign debt, following the initial contributions of Aguiar and Gopinath (2006) and Arellano (2008), studies sovereign spreads, debt capacity and welfare from a positive and normative point of view. The focus is usually on Markov equilibria on payoff relevant state variables and and hence defaults can only be consequence of bad fundamentals. Our paper shares with this strand of the literature the focus on a model along the lines of Eaton and Gersovitz (1981) but rather than characterizing a particular equilibrium, we study predictions across all equilibria. In addition, we provide a full characterization of the set of equilibrium and conditions for equilibrium multiplicity that are novel in the literature. The second direction, focuses on equilibrium multiplicity, and in particular, in self fulfilling debt crises. The seminal contributions are Calvo (1988) and Cole and Kehoe (2000). Our paper studies multiplicity in an alternative setup, the one of Eaton and Gersovitz (1981);[5] the crucial difference between the setting in Cole and Kehoe (2000) and the one in Eaton and Gerso-

Bajari et al., 2010) and incomplete (see for example De Paula and Tang, 2012) information.

[4]Moment conditions that yield inequality constraints, as observable implications of equilibria, have spurred a literature in Econometrics that studies inference and consistency of structural estimates that are based on moment inequalities (see for example, Chernozhukov et al. 2007, Beresteanu et al. 2011, Bugni, 2010, Romano and Shaikh, 2010), or estimates structural parameters in games with multiple equilibria (see for example Ciliberto and Tamer, 2009) among others. Identification of structural parameters is also a part of a much larger literature on partial identification in Econometrics (see for example Tamer, 2010 for a recent review).

[5]A recent exception that studies multiplicity in a model as in Eaton and Gersovitz (1981) is Stangebye (2018). Auclert and Rognlie (2016) find necessary and sufficient conditions for uniqueness in a model as in Eaton and Gersovitz (1981). Recent contributions to the strand of the literature that studies defaults due to fundamentals, among many others, are Bianchi et al. (2017), Chatterjee and Eyigungor (2015), Hatchondo et al. (2016), Pouzo and Presno (2016), Arellano and Bai (2014), Arellano and Bai (2017), Ottonello and Perez (2018), Aguiar et al. (2017), and Sanchez et al. (2018). Recent contributions to the strand that studies equilibrium multiplicity, following Calvo (1988) and Cole and Kehoe (2000), are Lorenzoni and Werning (2013), Bocola and Dovis (2016), Aguiar et al. (2017), Corsetti and Dedola (2016), Roch and Uhlig (2018), and Ayres et al. (2018). See Aguiar and Amador (2013) for a comprehensive review.

vitz (1981) is that in the latter the government issues debt (with commitment) and then the price is realized, changing the source of equilibrium multiplicity. Our contribution to this strand of the literature is that by providing sufficient conditions for equilibrium multiplicity in a model as in Eaton and Gersovitz (1981) we show that once we introduce coordination devices, under the right parametric assumptions, coordination failures are a robust feature in models of sovereign lending.

Outline. The paper is structured as follows. Section 2 introduces the model. Section 3 characterizes equilibrium consistent outcomes. Section 4 discusses the characterization of equilibrium consistent outcomes when there are correlating devices available after debt is issued. Section 5 spells out the general model and states the main results of the paper in this setup. Section 6 concludes.

2 A Dynamic Policy Game

Our model of sovereign debt follows Eaton and Gersovitz (1981). Time is discrete and denoted by $t \in \{0, 1, 2, ...\}$. A small open economy receives a stochastic stream of income denoted by y_t. Income follows a Markov process with c.d.f. denoted by $F(y_{t+1} \mid y_t)$. The government is benevolent and seeks to maximize the utility of the households. It does so by selling bonds in the international bond market. The household evaluates consumption streams according to

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

where $\beta < 1$ and u is increasing and strictly concave. The sovereign government issues short term debt at a price q_t. The budget constraint is

$$c_t = y_t - b_t + q_t b_{t+1}.$$

There is limited enforcement of debt. Therefore, the government will repay only if it is more convenient to do so. We assume that after a default the government remains in autarky forever after but there are not direct output costs of default. Furthermore, we also assume that the government cannot save

$$b_{t+1} \geq 0.$$

The assumption of no savings, which may implicitly capture political economy constraints that make it difficult for governments to save as modeled by Amador (2013), in addition to the assumption of no direct costs of default, are sufficient to guarantee that autarky is an equilibrium. The idea is that, if the government cannot save, and there are no output costs of default, if the government expects a zero bond price for its debt now and in every future period, then it will default its debt. To guarantee multiplicity we need to introduce conditions to guarantee that there is at least another equilibrium has a positive debt capacity. In our paper, this equilibrium with a positive price of debt is the Markov equilibrium that is usually studied in the literature of sovereign debt.[6]

Lenders. There is a competitive fringe of risk neutral investors that discount the future at a rate of $r > 0$. This discount rate, and the possibility of default, imply that the price of the bond is given by

$$q_t = \frac{1-\delta_t}{1+r}$$

where δ_t if the default probability on bonds b_{t+1} issued at date t.

Timing. The sequence of events within a period is as follows. In period t, the government enters with b_t bonds that it needs to repay. Then income y_t is realized. The government then has the option to default $d_t \in \{0, 1\}$. If the government does not default, the government runs an auction of face value b_{t+1}. Then, the price of the bond q_t is realized. Finally, consumption takes place, and is given by $c_t = y_t - b_t + q_t b_{t+1}$. If the government decides to default, then consumption is equal to income, $c_t = y_t$. The same is true if the government has ever defaulted in the past. We adopt the convention that if $d_t = 1$ then $d_{t'} = 1$ for all $t' \geq t$.

Histories, Strategies, and Outcomes. A *history* is a vector $h^t = (h_0, h_1, ..., h_{t-1})$, where $h_t = (y_t, d_t, b_{t+1}, q_t)$ is the the outcome of observable variables of the stage game at time t. A partial history is an initial history h^t concatenated with a history of the stage game at period t. For example, $\left(h^t, y_t\right)$ is the history after which the government must choose policies (d_t, b_{t+1}). The set of all partial histories is denoted by $\mathcal{H}$. We label as $\mathcal{H}_g \subset \mathcal{H}$ the partial histories where the policy maker has to choose policies. Likewise, $\mathcal{H}_m \subset \mathcal{H}$ is

[6] As we discuss in the Online Appendix C, and shown in Auclert and Rognlie 2016, no savings, $b_{t+1} \geq 0$, is a necessary conditions for equilibrium multiplicity. One of the contributions in our paper is to show that, no savings, plus a set of parametric conditions are sufficient for equilibrium multiplicity. Another paper studying multiplicity in the Eaton and Gersovitz (1981) setup is Stangebye (2018). The setup in the latter differs from us since there is long term debt and there are direct costs of default. The paper focuses on numerical results.

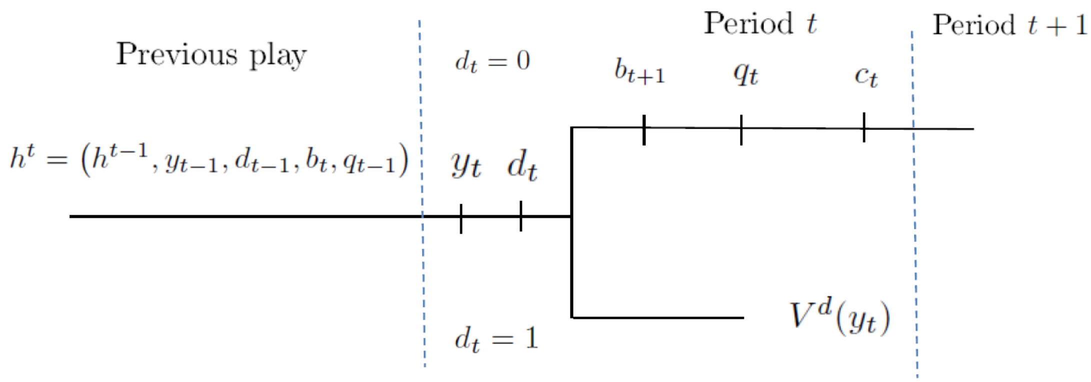

Figure 1: The figure summarizes the timing and the construction of histories.

the set of partial histories where the market plays; for example, $h_m^{t+1} = (h^t, y_t, d_t, b_{t+1})$. A policy maker's strategy is a function $\sigma_g(h^t, y_t) = (d_t, b_{t+1})$ for all histories. A rational expectation strategy for the market is a pricing function $q_m(h^t, y_t, d_t, b_{t+1})$ for all histories. Denote by Σ the set of strategies for the government and the market. For a strategy profile $\sigma = (\sigma_g, q_m)$ we write $V(\sigma \mid h)$ for the continuation expected utility, after history h, of the representative consumer if agents play according to profile σ. For any strategy profile $\sigma \in \Sigma$, we define the continuation at $h^t \in \mathcal{H}_g$

$$V(\sigma \mid h^t) = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \beta^s \left[(1-d_s) u(y_s - b_s + q_s b_{s+1}) + d_s u(y_s) \right] \right\}$$

where (y_s, d_s, b_{s+1}, q_s) are generated by the strategy profile σ.

Equilibrium. A strategy profile $\sigma = (\sigma_g, q_m)$ constitutes a *subgame perfect equilibrium (SPE)* if and only if, for all partial histories $h^t \in \mathcal{H}_g$

$$V(\sigma \mid h^t) \geq V(\sigma_g', q_m \mid h^t) \text{ for all } \sigma_g' \in \Sigma_g, \tag{2.1}$$

and for all histories $h_m^{t+1} = (h^t, y_t, d_t, b_{t+1}) \in \mathcal{H}_m$

$$q_m\left(h_m^{t+1}\right) = \frac{1}{1+r} \int (1 - d^{\sigma_g}(h^{t+1}, y_{t+1}) dF(y_{t+1} \mid y_t). \tag{2.2}$$

That is, the strategy of the government is optimal given the pricing strategy of the lenders $q_m(\cdot)$; likewise, $q_m(\cdot)$ is consistent with the default policy generated by σ_g. The set of all subgame perfect equilibria is denoted as $\Sigma^* \subset \Sigma$.

Equilibrium Consistency. We now introduce the concept of equilibrium consistency. Given a SPE profile $\sigma = (\sigma_g, q_m)$, we define its *equilibrium path* $x(\sigma)$ as a sequence of measurable functions $x(\sigma) = \left(d_t^{\sigma_g}(y^t), b_{t+1}^{\sigma_g}(y^t), q_t^{q_m}(y^t)\right)_{t\in\mathbb{N}}$ that are generated by following the profile σ. The outcomes in a particular period are defined as $x_t(\cdot) = \left(d_t^{\sigma_g}(\cdot), b_{t+1}^{\sigma_g}(\cdot), q_t^{q_m}(\cdot)\right)$, where $x_t(\cdot)$ is function of the realization of income history y^t. A history $h \in \mathcal{H}$ is *equilibrium consistent* if and only if it is on some equilibrium path $x = x(\sigma)$, for some SPE profile σ. Or, in other words, a history h^t is equilibrium consistent if we can find at least some equilibrium σ that explains the data.[7]

What Follows. The main question we would like to answer in our paper. Suppose that an outsider observes the history h^t. Is there a subgame perfect equilibrium profile that could have generated history h^t? If so, which are the possible continuation histories after observing h^t? In particular, given equilibrium histories h^t, which outcomes x_t are part of a continuation equilibrium?

3 Equilibrium Consistent Outcomes

This section discusses the main result of the paper, which is the characterization of equilibrium consistent outcomes. We work with the baseline case where income is a continuous random variable, as in Eaton and Gersovitz (1981). In subsection 3.1, we start by characterizing the set of equilibrium values and prices. Then, in subsection, 3.2, we state and describe our main result. Finally, in subsection 3.3, we apply our main result to obtain predictions for bond prices across all equilibria.

3.1 Equilibrium Prices, Continuation Values.

For any history h_m^{t+1} we define the highest and lowest prices equilibrium prices as:

$$\overline{q}(h_m^{t+1}) := \max_{\sigma\in\Sigma^*(h_m^{t+1})} q_m\left(h_m^{t+1}\right)$$

$$\underline{q}(h_m^{t+1}) := \min_{\sigma\in\Sigma^*(h_m^{t+1})} q_m\left(h_m^{t+1}\right).$$

[7] This definition will be instrumental in finding the defining conditions of equilibrium paths, by providing a recursive representation. A history is part of an equilibrium path if and only if the history up to $t-1$ is part of an equilibrium path and the partial history at time t is also consistent with it.

In the Online Appendix, Section C, we describe necessary and sufficient conditions for equilibrium multiplicity.[8] In particular, we show that the worst SPE price is zero and the best SPE price is the one for the Markov equilibrium that is characterized in the literature on sovereign debt, such as in Arellano (2008) and Aguiar and Gopinath (2006), where coordination failures do not have a role. The lowest price $\underline{q}(h_m^{t+1})$ is attained by using a fixed strategy for all histories, and the government will obtain the utility level of autarky. Thus, the lowest price is associated with the worst equilibrium, in terms of welfare. Likewise, the highest price $\overline{q}(h_m^{t+1})$ is associated with a, different, fixed strategy for all histories (the maximum is attained by the same σ for all h_m^{t+1}) and delivers the highest equilibrium level of utility for the government. Thus, the highest price is associated with the best equilibrium in terms of welfare. The autarky utility (conditional on defaulting) is given by:

$$V^d\left(y\right) \equiv u\left(y\right) + \beta \mathbb{E}_{y'|y} V^d(y'). \tag{3.1}$$

The continuation utility (conditional on not defaulting) of choice b' given bonds (b, y) is

$$\overline{V}^{nd}\left(b, y, b'\right) = u\left(y - b + b'\overline{q}\left(y, b'\right) b'\right) + \beta \overline{\mathbb{V}}\left(y, b'\right), \tag{3.2}$$

where $\overline{q}\left(y, b'\right)$ is the bond price schedule under the best continuation equilibrium (the Markov equilibrium that we just characterized), if $y_t = y$ and the bonds to be paid tomorrow are $b_{t+1} = b'$. Denote by $\overline{V}^{nd}\left(b, y\right) = \max_{b' \geq 0} \overline{V}^{nd}\left(b, y, b'\right)$. Finally, the continuation value of the best equilibrium, starting with income y and bonds b', is given by:

$$\overline{\mathbb{V}}(y, b') = \mathbb{E}_{y'|y}\left[\max\left\{\overline{V}^{nd}(b', y'), V^d(y')\right\}\right]. \tag{3.3}$$

3.2 Main Result

Suppose that, thus far, we have observed $h_m^{t-1} = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$ an equilibrium consistent history (where the price at time t has not yet been realized), and we want to characterize the set of *shifted* outcomes $x_{t,m} = \left(q_{t-1}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right)$ that are consistent with this

[8] There are two points worth noting. First, our analysis may be of independent interest, because we describe conditions under which there are multiple Markov equilibria in a sovereign debt model that follows Eaton and Gersovitz (1981), a framework that has been widely adopted in the literature. The importance of this result is that it opens up the possibility of confidence crises in a class of models that are usually utilized to study crises that are due to bad fundamentals. Thus, confidence crises are not necessarily a special feature of the timing in Calvo (1988) and Cole and Kehoe (2000) but rather a robust feature of most models of sovereign debt. Second, given our assumptions of no savings and no direct costs of default characterizing the equilibrium set is relatively straightforward. This will not be the case in Sections 4 and 5, where we will need to, first, characterize the equilibrium set, developing a procedure in the spirit of Abreu (1988); Abreu et al. (1990); Atkeson (1991), and then use this characterization to pin down equilibrium consistent outcomes.

history.[9] Proposition 1 provides a full characterization of the set of equilibrium consistent outcomes.

Proposition 1. *Suppose that $h_m^{t-1} = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$ is an equilibrium consistent history, with no default so far. Then, $x_{t,m} = \left(q_{t-1}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right)$ is equilibrium consistent with h_m^{t-1} if and only if the following conditions hold:*

a. The price is consistent with the default policy:

$$q_{t-1} = \frac{\mathbb{E}_{y_t|y_{t-1}}\left(1 - d_t(y_t)\right)}{1+r}; \tag{3.4}$$

b. Incentive compatibility for the government:

$$(1 - d(y_t))\left[u(y_t - b_t + \overline{q}(y_t, b_{t+1})b_{t+1}) + \beta\overline{\mathbb{V}}(y_t, b_{t+1})\right] + d(y_t)V^d(y_t) \geq V^d(y_t); \tag{3.5}$$

c. Promise keeping constraint:

$$\begin{aligned} \beta\mathbb{E}_{y_t|y_{t-1}}\left[(1 - d_t(y_t))\overline{V}^{nd}\left(b_t, y_t, b_{t+1}(y_t)\right)\right] + \beta\mathbb{E}_{y_t|y_{t-1}}\left[d_t(y_t)V^d\left(y_t\right)\right] \geq \\ \left[u\left(y_{t-1}\right) - u\left(y_{t-1} - b_{t-1} + q_{t-1}b_t\right)\right] + \beta\mathbb{E}_{y_t|y_{t-1}}V^d(y_t). \end{aligned} \tag{3.6}$$

Proof. See Appendix A. □

If conditions (a) through (c) hold, we simply write

$$\left(q_{t-1}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right) \in \mathbb{ECO}\left(b_{t-1}, y_{t-1}, b_t\right),$$

where $\mathbb{ECO}$ stands for "equilibrium consistent outcomes".

There are several points worth noting. First, note that the conditions (3.4) and (3.5) in Proposition 1 characterize the set of SPE outcomes. Condition (3.4) states that the price q_{t-1} needs to be consistent with the default policy $d_t(\cdot)$. Condition (3.5) states that the policy $d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)$ is implementable in an SPE if it is incentive compatible when the policy is rewarded with the best equilibrium and a deviation is punished with the worst equilibrium. The argument for the proof builds but also modifies the one on Abreu (1988).

[9] An outcome in period t is given by $x_t = \left(d_t^x\left(\cdot\right), b_{t+1}^x\left(\cdot\right), q_t^x\left(\cdot\right)\right)$ which includes the policies and prices of period t. $x_{t,m}$ represents the policies of period t but the prices of period $t-1$. The focus in $x_{t,m}$, in contrast to that of x_t, simplifies the characterization of equilibrium consistent outcomes.

Second, note that equilibrium consistent outcomes are characterized by an additional condition, (3.6), which is the main contribution of this paper. This condition decribes how past observed history (if *assumed* to be generated by an *equilibrium* strategy profile) introduces restrictions on the set of equilibrium consistent policies. In our setting, condition (3.6) guarantees that the government's no default decision at $t-1$ was optimal. That is, on the path of some SPE profile $\hat{\sigma}$, the incentive compatibility (IC) constraint from government's utility maximization in $t-1$ is that the value of staying on path, $u\left(c_{t-1}\right)+\beta V\left(\hat{\sigma} \mid h^{t}\right)$, is greater than or equal to the value of a deviation $u\left(y_{t-1}\right)+\beta \mathbb{E}_{y_t|y_{t-1}} V^{d}(y_t)$. Note that $V\left(\hat{\sigma} \mid h^{t}\right)$ is the continuation value of the equilibrium, as defined before. One interpretation of this incentive compatibility constraint, is that the net present value (with respect to autarky) that the government expects from not defaulting must be greater (for the past choice to be optimal) than the opportunity cost of not defaulting: $u\left(y_{t-1}\right)-u\left(c_{t-1}\right)$. This must be true for *any* SPE profile that could have generated h_m^{t-1}.

The intuition regarding why (3.6) is *necessary* for equilibrium consistency is as follows. Note that, if incentive compatibility at $t-1$ holds for some equilibrium, it also holds for the case the in which continuation equilibrium is actually the best (continuation) equilibrium. Denote by $\hat{q}_t = \hat{q}_t\left(h^t, y_t, d_t, b_{t+1}\left(y_t\right)\right)$. For any equilibrium consistent policy $\left(d\left(\cdot\right), b'\left(\cdot\right)\right)$, it has to be the case that:

$$\mathbb{E}_{y_t|y_{t-1}}\left[(1-d_t(y_t))\overline{V}^{nd}\left(b_t, y_t, b_{t+1}(y_t)\right)\right]+\mathbb{E}_{y_t|y_{t-1}}\left[d_t(y_t)V^d\left(y_t\right)\right] \geq \tag{3.7}$$

$$\mathbb{E}_{y_t|y_{t-1}}\left[(1-d_t(y_t))\left(u(y_t-b_t+b_{t+1}(y_t)\hat{q}_t)+\beta V(\hat{\sigma}\mid h^{t+1})\right)\right]+\mathbb{E}_{y_t|y_{t-1}}\left[d_t(y_t)V^d\left(y_t\right)\right]$$

where the right hand side of equation (3.7) is equal to $V\left(\hat{\sigma} \mid h^t\right)$. From incentive compatibility in $t-1$ and (3.7), we obtain the following:

$$\mathbb{E}_{y_t|y_{t-1}}\left[(1-d_t(y_t))\overline{V}^{nd}\left(b_t, y_t, b_{t+1}(y_t)\right)\right]+\beta\mathbb{E}_{y_t|y_{t-1}}\left[d_t(y_t)V^d\left(y_t\right)\right] \geq \tag{3.8}$$

$$\left[u\left(y_{t-1}\right)-u\left(y_{t-1}-b_{t-1}+q_{t-1}b_t\right)\right]+\beta\mathbb{E}_{y_t|y_{t-1}}V^d(y_t).$$

This is exactly condition (3.6). Therefore, if the policies do not satisfy (3.6), then there is no SPE that can generate the history h_m^{t-1}; in other words, there is no SPE consistent with h_m^{t-1} with policies $\left(d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right)$ for period t.

We also show that this condition is *sufficient*, so if $\left(d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right)$ satisfies the conditions (3.4), (3.5), and (3.6), we can always find at least one SPE profile $\hat{\sigma}$ that would generate $x_{t,m}$ on its equilibrium path. Even after a long history of data, the sufficient statistics to forecast the outcome $x_{t,m}$ are $\left(b_{t-1}, b_t, y_{t-1}\right)$. Thus, effectively $\mathbb{ECO}(h_{-}^{t}) =$

$\mathbb{ECO}(b_{t-1}, y_{t-1}, b_t)$. This result may seem surprising, but it is a direct consequence of robustness of the outside observer is expressed. In particular, because income y is a continuous random variable, any promises (in terms of expected utility) that rationalized past choices are "forgotten" each period; the reason is that the outside observer needs to take into account that the promises *could* have been realized in states that did not occur.

Third, and finally, note that even though the outside observer is using just a small fraction of the history to place restrictions on the observable outcomes, the set of equilibrium consistent outcomes exhibits history dependence beyond that of the set of SPE. In particular, the set of equilibrium consistent outcomes is a function of the variables (b_{t-1}, y_{t-1}, b_t). Thus, there is a role for past actions in placing restrictions over observable outcomes. We view this result as an application of the revealed preference arguments in Jovanovic (1989) and Pakes et al. (2015) to dynamic games.[10]

3.3 Equilibrium Consistent Prices

The question that we would like to answer now is the following: Given an observed history h_m^{t-1}, which are the possible continuation prices? Aided by the characterization of equilibrium consistent outcomes in Proposition 1 we will characterize the set of equilibrium debt prices that are consistent with the observed history $h_m^{t-1} = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$. This set of prices comes from the restrictions of equilibrium for the observable variables. There are two objects of interest: the highest and the lowest equilibrium consistent prices.

Prices. The *highest* equilibrium consistent price solves

$$\overline{q}\left(h_m^{t-1}\right) = \max_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q}$$

subject to

$$\left(\hat{q}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right) \in \mathbb{ECO}\left(b_{t-1}, y_{t-1}, b_t\right).$$

This price is the one for the (best) Markov Equilibrium that we characterized in the Online Appendix Section C, where after a default, the government is forever in autarky. Note that the expected value of the incentive compatibility constraint (3.5) is the expected value of the option to default $\overline{\mathbb{V}}(y_t, b_{t+1})$ for the best equilibrium, which is given by equation (3.3).

[10] In regard to the particular model we are analyzing, the repayment of debt affects future prices; this implication of repayment does not appear in the quantitative literature for sovereign debt that follows Eaton and Gersovitz (1981) as in Arellano (2008) and Aguiar and Gopinath (2006). In these papers, the fact that a country has just repaid a large quantity of debt, does not affect the future prices that the country will obtain.

The promise-keeping constraint will not be binding (generically) for the best equilibrium (given that the country did not default). For these two reasons, the best equilibrium consistent price is the one obtained with the default and bond policy that maximize the value of the option. Thus, $\overline{q}\left(h_m^{t-1}\right) = \overline{q}\left(y_{t-1}, b_t\right)$.

The *lowest* equilibrium consistent price solves

$$\underline{q}\left(h_m^{t-1}\right) = \min_{\left(\hat{q}, d_t(\cdot), b_{t+1}(\cdot)\right)} \hat{q}$$

subject to

$$\left(\hat{q}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right) \in \mathbb{ECO}\left(b_{t-1}, y_{t-1}, b_t\right).$$

Characterizing this price is slightly more challenging. Note that the lowest SPE price is zero because default is implementable after any history if we do not take into account the promise-keeping constraint (3.6). However, we will show that the lowest equilibrium consistent price is positive, for every equilibrium history. Furthermore, because the set of equilibrium consistent outcomes after history h_m^t depends only on $\left(b_{t-1}, y_{t-1}, b_t\right)$, it holds that the lowest equilibrium consistent price is history dependent; $\underline{q}\left(h_m^{t-1}\right) = \underline{q}\left(b_{t-1}, y_{t-1}, b_t\right)$.

Proposition 2 establishes the main result of this subsection: a characterization of $\underline{q}$ that is a solution for a (convex) minimization program, which can be reduced to a one equation/one variable problem.

Proposition 2. *Suppose that h_m^{t-1} is equilibrium consistent and that not defaulting was feasible under the best continuation equilibrium; i.e. $\overline{V}^{nd}\left(b_{t-1}, y_{t-1}, b_t\right) > V^d\left(y_{t-1}\right)$. Then, there exists a constant $\gamma = \gamma\left(b_{t-1}, y_{t-1}, b_t\right) \geq 0$ such that $\underline{d}\left(y'\right) = 0 \iff \overline{V}^{nd}\left(b_t, y_t\right) \geq V^d\left(y_t\right) + \gamma$ for all $y_t \in Y$; therefore, the lowest equilibrium consistent price is given by*

$$\underline{q}\left(b_{t-1}, y_{t-1}, b_t\right) = \frac{\mathbb{E}_{y_t|y_{t-1}}\left(1 - \underline{d}\left(y_t\right)\right)}{1+r}.$$

Proof. See Appendix A. □

The proof is in the appendix. Here, we provide a brief discussion of the argument. First, note that by choosing the bond policy of the best equilibrium, all of the constraints imposed by equilibrium consistency are relaxed because the value of not defaulting increases. Therefore, finding the lowest ECO price will amount to finding the default policy that yields the lowest price that is consistent with equilibrium. Second, note that the promise-keeping constraint needs to be binding. If not, the minimization problem has as its only constraint the incentive compatibility constraint, and the minimum price is zero (with a policy of default in every state). However, if the price is zero, then the promise keeping constraint will not be satisfied. Third, note that the incentive compatibility constraint will not be binding. Intuitively, imposing a default is not costly in terms of incentives, and for the lowest equilibrium consistent price, we want to impose default in as many states as possible.

Considering these observations, note that the trade-off of the default policy for the lowest price will be: imposing defaults in more states will lower the price at the expense of a tighter promise keeping constraint. This condition pins down the states where the government defaults; as many defaults as possible, but not so many that achieving no default in the previous period was not worth the effort. This result implies that the policy is pinned down by $\underline{d}\,(y_t) = 0$ if and only if $\overline{V}^{nd}\,(b_t, y_t) \geq V^d\,(y_t) + \gamma$ where γ is a constant to be determined.

Note how default policies are tilted deferentially in the best and worst continuation equilibria. For the best equilibrium default policy at t, it holds that $d(y_t) = 0$ if and only if $\overline{V}^{nd}(b_t, y_t) \geq V^d(y_t)$. On the other hand, the lowest equilibrium consistent price is $\overline{V}^{nd}(b_t, y_t) \geq V^d(y_t) + \gamma$, where γ is the constant that, as we will see below, solves a one equation in one unknown system and depends on (b_{t-1}, y_{t-1}, b_t). The default policy is shifted to create more defaults and to lower the price; the number of defaults is limited, however, so that the promise-keeping is satisfied (i.e., if not, we cannot rationalize previous choices). Equilibrium consistent outcomes uncover a novel tension that is not present in SPE. For a particular history h^t_-, implementing default is not costly because it is always as good as the worst equilibrium. However, implementing default today lowers the prices that the government expected in the past and makes it harder to rationalize a particular history.

Next, we discuss the final piece: how we obtain γ? Define $\Delta^{nd}\,(b_{t+1}, y_{t+1}) = \overline{V}^{nd}\,(b_t, y_t) - V^d\,(y_t)$. This constant, γ, solves a single equation and is the minimum value such that the promise keeping constraint holds with equality, with the optimal bond policy, which is evaluated at the best continuation; i.e:

$$\beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}\left(\Delta^{nd} \mid y_{t-1}\right) - u\,(y_{t-1}) + u\left(y_{t-1} - b_{t-1} + b_t \frac{1 - \hat{F}\,(\gamma \mid y_{t-1})}{1+r}\right) = 0.$$

Comparative Statics. The next result, Corollary 1, describes how the set of equilibrium consistent prices, $\left[\underline{q}, \overline{q}\right]$ changes with the history of play and follows directly from Propositions 1 and 2. After presenting the corollary and discussing its intuition, we provide a numerical illustration of the results in this Section.

Corollary 1. *Let $\underline{q}(b_{t-1}, y_{t-1}, b_t)$ be the lowest $\mathbb{ECO}(b_{t-1}, y_{t-1}, b_t)$ price after history h_m^{t-1}. The following holds: (a) $\underline{q}(b_{t-1}, y_{t-1}, b_t)$ is decreasing in b_t; (b) $\underline{q}(b_{t-1}, y_{t-1}, b_t)$ is increasing in b_{t-1}; and (c) For every equilibrium (b_{t-1}, y_{t-1}, b_t), $-b_{t-1} + \underline{q}(b_{t-1}, y_{t-1}, b_t)\, b_t \leq 0$; if income is i.i.d., then $\underline{q}$ is decreasing in y_{t-1}, and so is the set of equilibrium consistent prices $[\underline{q}(b_{t-1}, y_{t-1}, b_t), \overline{q}(y_{t-1}, b_t)]$.*

Proof. See Appendix A. □

First, note that the lowest equilibrium consistent price is decreasing in the amount of debt issued b_t. The intuition is that higher amounts of debt issued imply a more relaxed promise-keeping constraint. In other words, the past choices of the government could be rationalized with a lower price for the debt b'. The opposite intuition holds for b_{t-1}; if the country just repaid a large amount of debt (i.e., made an effort to repay the debt), then the past choices are rationalized by using higher prices. Second, note that a positive capital inflow obtained at the the lowest equilibrium consistent price would imply that $u(y_{t-1}) - u\left(y_{t-1} - b_{t-1} + \underline{q}(b_{t-1}, y_{t-1}, b_t)\, b_t\right)$ is negative. Intuitively, the country is not making any effort to repay the debt. Therefore, it need not be the case that the country expects high prices for debt in the next period. Mathematically, when there is a positive capital outflow with the lowest equilibrium consistent price, γ is infinite. This result implies that $\frac{1-\hat{F}(\gamma)}{1+r} = \underline{q}(b_{t-1}, y_{t-1}, b_t) = 0$, which contradicts a positive capital inflow. Finally, because there are no capital inflows at the lowest equilibrium consistent price, repaying debt at this price will become more costly for a lower realization of income y_{t-1}; this due to the concavity of the utility function. Mathematically, because of concavity, $u(y_{t-1}) - u\left(y_{t-1} - b_{t-1} + \underline{q}(b_{t-1}, y_{t-1}, b_t)\, b_t\right)$ is increasing as income decreases, and therefore, the promise-keeping constraint tightens as income decreases.[11]

A Quantitative Illustration. We now numerically solve for the equilibrium consistent prices. The process for log output is given by $\log y_t = \mu + \rho_y \log y_{t-1} + \sigma_y \epsilon_t$ where $\mu = 0.75$, $\sigma_y = 0.3025$, and $\rho_y = 0.0945$. The risk free interest rate is set to $r = 0.017$. The utility function is $u(c) = \frac{c^{1-\gamma_{RRA}}}{1-\gamma_{RRA}}$, the coefficient of relative risk aversion is $\gamma_{RRA} = 2$, and

[11]There are three important points here. First, the observation regarding concavity noted in the last sentence is used often in the literature on sovereign debt. For example, to show that default occurs in bad times, as in Arellano (2008), or to show the monotonicity of bond policies with respect to debt, as in Chatterjee and Eyigungor (2012). Second, the change in this expression will depend on the sign of $u(y_{t-1}) - u\left(y_{t-1} - b_{t-1} + \frac{1-\hat{F}(\gamma)}{1+r} b_t\right)$, which is positive because there are no capital inflows with the lowest equilibrium consistent price. Third, note that, in the non i.i.d. case, this property will not hold, because, even though the burden of repayment is higher, the value of repayment in terms of the continuation value can be increasing.

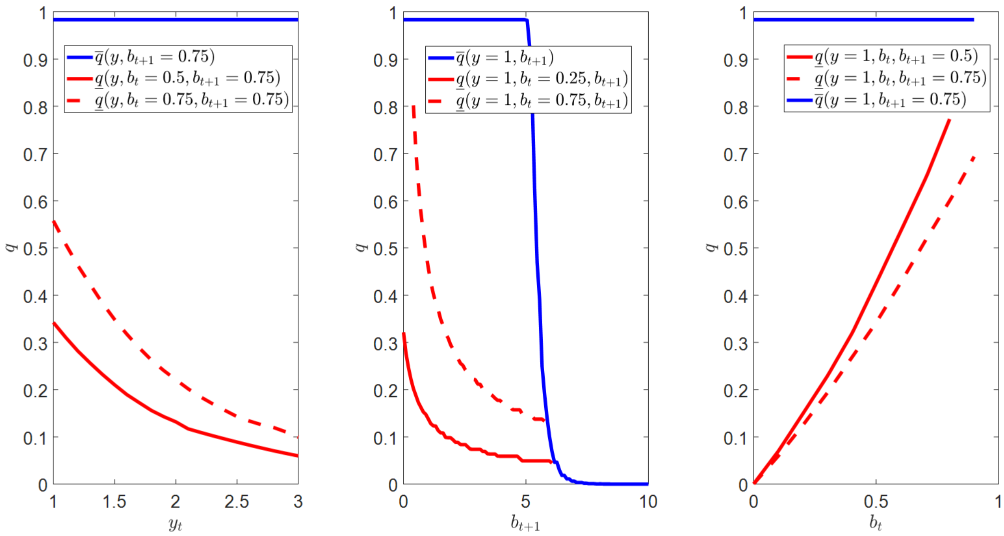

Figure 2: This figure plots equilibrium consistent prices $\overline{q}$ and $\underline{q}$. We describe the comparative statistics after history h^t_m.Thus, the relevant state variables are (b_t, y_t, b_{t-1}).

the discount factor $\beta = 0.953$.[12]Figure 2 depicts the numerical results.

As we discussed before, the best equilibrium, $\overline{q}$, coincides with the equilibrium studied in the quantitative literature of sovereign debt, such as Arellano (2008). We plot the best equilibrium in blue and the lowest equilibrium consistent price in red. As clearly shown in the Figure, the best equilibrium for low levels of debt is risk-free. As we increase the level of debt, the price drops, and the price drop is sharp, as it is in most models with short-term debt.

The lowest equilibrium consistent price $\underline{q}(b_{t-1}, y_{t-1}, b_t)$ is computed using Proposition 2. Note that the comparative statistics that we specified in the Corollary 1 clearly emerge in Figure 2. First, in the left panel, when the government repays debt $b_t = 0.5$ and issues $b_{t+1} = 0.75$, the lowest equilibrium consistent price decreases in the realization of income. This result occurs because the government repaid debt, but as we increase y_t,it did so under more favorable conditions. In addition, as one would expect, when the amount of debt repaid climbs to $b_t = 0.75$ and the amount of debt issued is still $b_{t+1} = 0.75$, the red dotted line dominates the red line. The lowest equilibrium consistent price is now higher. Finally, note that the best equilibrium price is constant through the realizations of income, because for those levels of debt, $b_{t+1} = 0.75$, default is not a concern. Second, in the right panel we observe that with debt repayment, b_t, we obtain the opposite: when

[12]We set the same parameters values for all the numerical exercises in this section, Section 4 and in the Online Appendix Section C.

the government repays a larger amount of debt, then the lowest equilibrium consistent price increases. This is the case for both $(y_t = 1, b_{t+1} = 0.50)$ and $(y_t = 1, b_{t+1} = 0.75)$.[13] The dotted line corresponds to a higher debt issuance, and as we just discussed, given a larger capital inflow, the prices are expected to be lower.

4 Equilibrium Consistent Distributions

In Section 3 we characterized equilibrium consistent outcomes and aided with this characterization we constructed bounds on equilibrium prices. These bounds are tight. Any price outside $\left[\underline{q}(b_{t-1}, y_{t-1}, b_t), \overline{q}(y_{t-1}, b_t)\right]$ occurs with probability zero. These tight predictions are a consequence of the special feature of the setup in Section 3 by which the actions of the government, d_{t-1}, b_t, and the prices they obtain for debt q_{t-1} are connected with a deterministic mapping. There are many settings in which one would think that there is not deterministic link between policies and outcomes. One example is sovereign borrowing. The government does not need to know what prices they will obtain given their policies. This is, in fact, a widely studied topic in models of sovereign borrowing at least since Calvo (1988). [14]

To break the deterministic mapping between policies and prices in our baseline model of Section 3, we introduce a sunspot between the moment in which the large player moves and the market reacts. In particular, we generalize the setup in Eaton and Gersovitz (1981) by adding a sunspot variable ζ_t after the government issues debt but before the price is realized.[15] As a consequence of the introduction of the sunspot, conditional on any single realization, the set of equilibrium consistent outcomes then coincides with the set of subgame perfect equilibria. That is, for any history of policies chosen by the

[13] This result may be contrasted with the result in Cole and Kehoe (2000). In their setting the potential for rollover crises induces the government to lower debt below a threshold that rules rollover crises out. Thus, the government's efforts have no effect in the short run, but payoff in the long run. In our model, an outside observer will witness that rollover crises are less likely immediately after an effort has been made to repay the debt.

[14] Theoretical models of sovereign debt that are prone to multiple equilibria are, for example, Cole and Kehoe (2000), Aguiar et al. (2017), Stangebye (2014) and Bocola and Dovis (2016). Another strand of the literature is Calvo (1988) and Lorenzoni and Werning (2013). Yet another strand is the work of Corsetti and Dedola (2016). Another example, of a weaker link, is models of monetary policy. There is a large literature on equilibrium indeterminacy in New Kenynesian Models that studies which rules guarantee that a unique equilibrium can be obtained. Equilibrium multiplicity breaks the link between the interest rate chosen by the central bank and the realizations of output and inflation. For models of monetary policy see for example Benhabib et al. (2001), Lubik and Schorfheide (2004) and Mertens and Ravn (2014).

[15] It is worth noting that adding a sunspot that is realized together with output adds nothing to the analysis. Effectively, the output could already be acting as a random coordination device. Thus, the interesting question is to add a sunspot variable after the bond issuance, but before the price is determined.

government, any equilibrium price can be observed; i.e., any price $q_{t-1} \in [0, \overline{q}(y_{t-1}, b_t)]$, that we characterize in section C of the Appendix.

But this raises a question: Does the fact that h_m^{t-1} is generated by an equilibrium place any restrictions over outcomes? At first, it looks like histories will have no bite in pinning down future outcomes. Surprisingly, as we will show in the main result of this section, Proposition 3, we will obtain history dependent predictions. However, these restrictions will be across *distributions* of debt prices.[16] The idea is that, for example, a distribution that puts probability one to a zero price cannot be an equilibrium distribution for any history; in particular if the government has repaid a positive amount of debt. Given these restrictions over probabilities, it is intuitive to conjecture that also the means and variances of distributions over prices, as well as other moments of the distributions, will be pinned down.[17]

What follows. In this section we do three things. First, in Subsection 4.1, we start by characterizing the best equilibrium continuation values for the government given a realization of prices . We already characterized the set of equilibrium values, and prices. However, for this section it will be useful to know the best continuation after a particular price realization (ex-post best continuation value). Second, in the main result of the section, Proposition 3, we characterize what we term as *equilibrium consistent distributions*,which are probability distributions over prices that are consistent with a SPE given history. This result parallels the main result in Section 3, Proposition 1. Third, aided by this characterization, as in the version of the model without sunspots, we explore the restrictions implied over observables of the assumption that the history is generated by some equilibrium, thus making Proposition 3 operational. In Proposition 4 we find bounds on the probability of a non-fundamental debt crises, where a crisis refers to an event where the realized price falls below a given threshold $\hat{q}$. In Propositions 5 and 6 we obtain bounds of the expected prices and their variance that hold across all equilibria. Finally, in Corollary 1, we compute comparative statistics for the set of equilibrium con-

[16]This is the main insight of Aumann (1987b) notion of Correlated equilibrium, where instead of characterizing the mapping between information and strategies, we can obtain directly constraints directly on equilibrium strategy distributions. The same principle also works in settings of incomplete information, as has been recently studied in Kamenica and Gentzkow (2011), Benoît and Dubra (2011) and for a general setting as the concept of Bayesian Correlated Equilibrium in Bergemann and Morris (2016).

[17]The importance of the bound on distributions over outcomes is that they will permit to obtain set identification of parameters. As we mentioned in the introduction, this paper relates to the previous findings on the observable implications of models with multiple equilibria; Jovanovic (1989) and Pakes et al. (2015). These implications over observables, often moment conditions, can be used to recover structural parameters of interest. As we mentioned before, our paper is the paper to derive testable implications of equilibrium without any restriction in the set of equilibrium strategies.

sistent distributions and show that the set is ordered according to first order stochastic dominance.

4.1 Ex-Post Best Continuation Value

The maximum continuation value function $\overline{v}(y_-, b, q_-)$ given bonds b, issued at a price q_-, when income is y_-, is defined as $\overline{v}(y_-, b, q_-) := \max_{\sigma \in \mathbf{\Sigma}^*(y_-, b)} V(\sigma \mid y_-, b, q_-)$. In Appendix D we show that this function can be computed as:

$$\overline{v}(y_-, b, q_-) = \max_{d(\cdot) \in [0,1]^Y} \mathbb{E}_{y'|y} \left[d(y) V^d(y) + (1 - d(y)) \overline{V}^{nd}(b, y) \right]$$

subject to

$$q_- = \frac{\mathbb{E}_{y|y_-}(1 - d(y))}{1 + r}.$$

We also show that $\overline{v}(y_-, b, q_-)$ is non-increasing in b, and non-decreasing and concave in q_-.[18] The fact that the function is non-decreasing in q follows from the fact that better prices are associated with better continuation equilibrium, as well as higher contemporaneous consumption (since $b_{t+1} \geq 0$). Concavity follows from the the fact that $\overline{v}(y_-, b, q_-)$ solves a linear programming problem. We use both properties to obtain sharper characterizations of the set of equilibrium consistent distributions and to obtain testable predictions.

4.2 Main Result: Equilibrium Consistent Distributions

As we just mentioned ζ_t denotes the sunspot that is realized after the government issues bonds b_{t+1}, but before the price q_t is determined; i.e, a sunspot is realized after h_m^t. The timeline is depicted in Figure 3. Without a loss of generality we assume that $\zeta_t \sim \text{Uniform}[0,1]$ i.i.d. over time.[19] At history $h_m^{t+1} = (h^t, y_t, d_t, b_{t+1})$, given an equilibrium strategy $\sigma = (\sigma^g, q^m)$, the associated equilibrium price distribution at t is defined by $\Pr(q_t \in A) =: \Pr(\zeta_t : q_t^\sigma(h_m^{t+1}, \zeta_t) \in A)$. Denote by $\mathbb{ECD}(h_m^{t+1})$ the set of equilibrium

[18] Note that the set of equilibrium strategies only depends on the initial bonds and the seed value of income $\mathbf{\Sigma}^*(y_-, b)$. In the case of i.i.d. income, then it would be the case that $\mathbf{\Sigma}^*(b)$. We relegate the details to Appendix D. We will use interchangeably, the notation $\overline{v}(y_-, b, q_-)$ or $\overline{v}(y, b', q)$, depending of what is more convenient.

[19] The reason that this assumption implies no loss of generality is a direct consequence of robustness: we will try to map all equilibria that can be contingent on the randomizing device, and hence as long as the random variable remains absolutely continuous, any time dependence in ζ_t can be replicated by time dependence on the equilibrium itself.

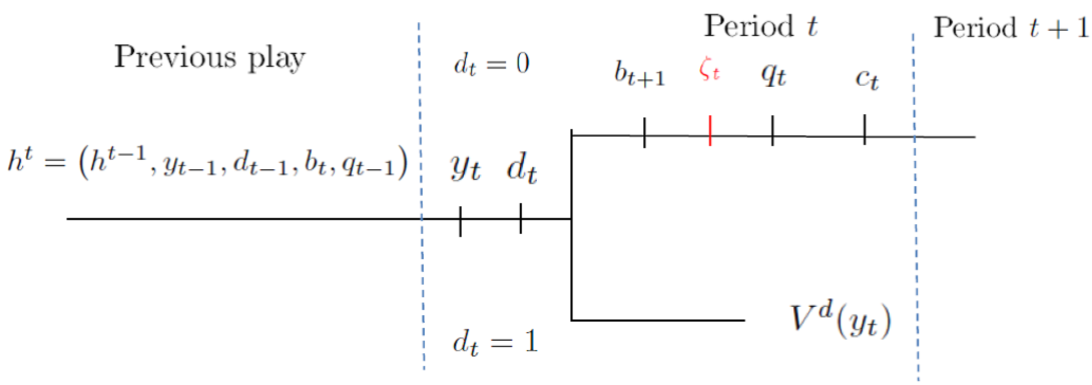

Figure 3: The figure summarizes the timing and the construction of histories in the case in which there is a sunspot. Now, we introduce a sunspot ζ_t after the government has issued debt b_{t+1} and before the price q_t has been realized.

price distributions from history consistent equilibria. The following proposition characterizes this set.

Proposition 3. *Suppose that $h_m^{t+1} = (h^t, y_t, , b_{t+1})$, with no default so far, is equilibrium consistent. Then, the distribution $Q \in \Delta(\mathbb{R}_+)$ is an equilibrium consistent price distribution; i.e. $Q \in \mathbb{ECD}(h_m^{t+1})$ if and only if: (a) $Q \in \Delta([0, \overline{q}(y_t, b_{t+1})])$ and (b) IC of the government:*

$$\int_0^{\overline{q}(y_t, b_{t+1})} [u(y_t - b_t + q b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, q)] \, dQ(q) \geq V^d(y_t). \tag{4.1}$$

Proof. See Appendix B. □

Condition (4.1) parallels conditions (3.5) and (3.6) in Proposition 1. There are some differences, though. First, and most important, we now characterize the *distributions* over prices that are consistent with a decision of defaulting or not d_t and a debt issuance debt b_{t+1}. Proposition 1 characterized the complete outcome $x_{t+1,m}$; prices q_{t-1}, as well as policies d_t and b_{t+1}. Here, Proposition 3, characterizes prices q_t given policies d_t, b_{t+1}. Second, now, the payoff for the government is an expectation with respect to a measure Q over prices q (to be more precise, q_t). This breaks the deterministic mapping between government decisions and market prices; in the model without sunspots, in equilibrium, the government knows the debt price they will obtain before deciding not to default and how much debt to issue.

Why is condition (4.1) necessary and sufficient? The idea of the proof is an extension of the argument that proves Proposition 1. Fix an equilibrium consistent distribution Q after history h_m^{t+1}. If we assume that h_m^{t+1} is on the equilibrium path of some SPE, then the government strategies, d_t and b_{t+1}, were optimal before the realization of the sunspot ζ_t. This implies that the government ex-ante preferred to pay the debt (i.e. $d_t = 0$) and issue bonds (b_{t+1}) rather than defaulting on the debt. If, after these decisions the price realized is q, the payoff for the government would be *at most* $u(y_t - b_t + qb_{t+1})$ plus the best ex-post continuation value $\beta\overline{v}(y_t, b_{t+1}, q)$. However, the government has no certainty regarding the price that will be realized for the debt issued. So, the government forms an expectation with respect to the "candidate" equilibrium consistent distribution Q. This expectation, and it's associated expected utility, the left hand side of condition (4.1), has to at least as good as defaulting; if not, then the government would have defaulted and would not be issuing debt. This precisely describes condition (4.1), which is necessary because if it were to be violated, then we could not construct promises that rationalize the past history h_m^t.[20] The idea of sufficiency, in other words the reason why we eliminate b_{t-1} and all the previous policies, again stems from the fact that both the output and the sunspot are non-atomic.[21] The particular history that followed h_m^{t-1} when b_{t-1} was chosen, the one with the particular realization of y_t, had zero probability of occurring. Thus, it could always have been the case that the payoffs that rationalized b_{t-1} and the previous policies were to be realized in a state that never materialized.

Finally, two points are worth noting. Note that $\mathbb{ECD}\left(h_m^{t+1}\right) = \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$; we only use the most recent history, as in Proposition 1. In addition, can we employ conditions (4.1) for the case without sunspots? Yes. Note that in the case without sunspots that we analyzed in the previous section, the condition for equilibrium consistency is the static payoff $u(y_t - b_t + qb_{t+1})$ plus the continuation value$\beta\overline{v}(y_t, b_{t+1}, q)$ has to be greater than or equal to $V^d(y_t)$. The lowest equilibrium consistent price that we characterized in Section 3 will be pinned down by this condition with equality.

4.3 Implications of Equilibria: Bounding Price Distributions

We now delve into the implications of Proposition 3 over observable variables; in particular, distributions over prices q_t. The first set of implications are over the probability of low prices. In particular, we characterize the maximum probability that a crisis will occur. Second, we provide bounds across all equilibria for the expectation of prices. Third, we

[20]One might wonder why we cannot rely on on the best continuation payoff $\overline{\mathbb{V}}(y_t, b_{t+1})$. This is because this payoff is associated with the best equilibrium price, and this is price need to be realized. The best possible payoff, after the price q is realized, is precisely $\overline{v}(y_t, b_{t+1}, q)$.

[21]Even if output where discrete, sunspots make shocks non-atomic, having the same effect as if we had absolutely continuous output shocks.

also provide bounds across all equilibria for the variance of distributions over prices.[22] Finally, to close this subsection, we study the comparative statistics for the set of equilibrium consistent distributions, $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$.

Probability of Crises and the Infimum Distribution. We would like to infer the maximum probability (across equilibria) that the government assigns to a price $\hat{q}$; i.e., a crisis. Formally, we define the function $\underline{Q}\left(\hat{q}\right)$ as:

$$\underline{Q}\left(\hat{q}; b_t, y_t, b_{t+1}\right) \equiv \max_{Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)} \Pr_Q\left(q \leq \hat{q}\right) \tag{4.2}$$

where $\Pr_Q\left(q \leq \hat{q}\right) := \int_0^{\hat{q}} dQ\left(q\right)$. Furthermore this bound will yield a necessary condition for a distribution to be an element in $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$. The following proposition summarizes the results.

Proposition 4. *Consider an equilibrium consistent history $h_m^{t+1} = \left(h^t, y_t, d_t = 0, b_{t+1}\right)$. (a) For any $\hat{q} \geq \underline{q}\left(b_t, y_t, b_{t+1}\right)$, $\underline{Q}\left(\hat{q}; b_t, y_t, b_{t+1}\right) = 1$. (b) For any $\hat{q} < \underline{q}\left(b_t, y_t, b_{t+1}\right)$ it holds that:*

$$\underline{Q}\left(\hat{q}; \cdot\right) = \frac{\overline{V}^{nd}\left(b_t, y_t, b_{t+1}\right) - V^d\left(y_t\right)}{V^d\left(y_t\right) - \left[u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right] + \overline{V}^{nd}\left(b_t, y_t, b_{t+1}\right) - V^d\left(y_t\right)} \tag{4.3}$$

Proof. See Appendix B. □

Lets us start with the case $\hat{q} \geq \underline{q}\left(b_t, y_t, b_{t+1}\right)$. The reason why $\underline{Q}\left(\hat{q}; b_t, y_t, b_{t+1}\right)$ is equal to one is intuitive. A probability distribution that places a probability equal to one over $\underline{q}\left(b_t, y_t, b_{t+1}\right)$ will be an equilibrium consistent distribution. In this case $\Pr_Q\left(q \leq \hat{q}\right)$ is going to be equal to one. Thus, the max over the equilibrium consistent distributions will be equal to one. The case in which $\hat{q} < \underline{q}\left(b_t, y_t, b_{t+1}\right)$ is not that simple. Proposition 4 finds the maximum ex-ante probability (before ζ_t is realized) of observing a prices q_t, lower than $\hat{q}$, and it will be less that one. The idea of the proof is as follows. To relax the IC constraint for the government, condition (4.1), as much as possible, we need to we do the following: we consider distributions that are binary and assign prices $\{\hat{q}, \overline{q}\}$, and assign the best continuation equilibria when $\overline{q}$ is realized and the best ex-post continuation equilibrium when $\hat{q}$ is realized. This distribution, that we label $\underline{Q}\left(\hat{q}\right)$ needs to be as good as defaulting. When we equalize the value of issuing debt with the distribution $\underline{Q}\left(\hat{q}\right)$ to the value of defaulting, it implies then that $\underline{Q}\left(\hat{q}\right)$ is given by (4.3).

[22] All of these bounds are independent of the nature of the sunspots (i.e. the distribution of sunspots, its dimensionality, and so on), in the same way as the set of correlated equilibria does not depend on the actual correlating devices.

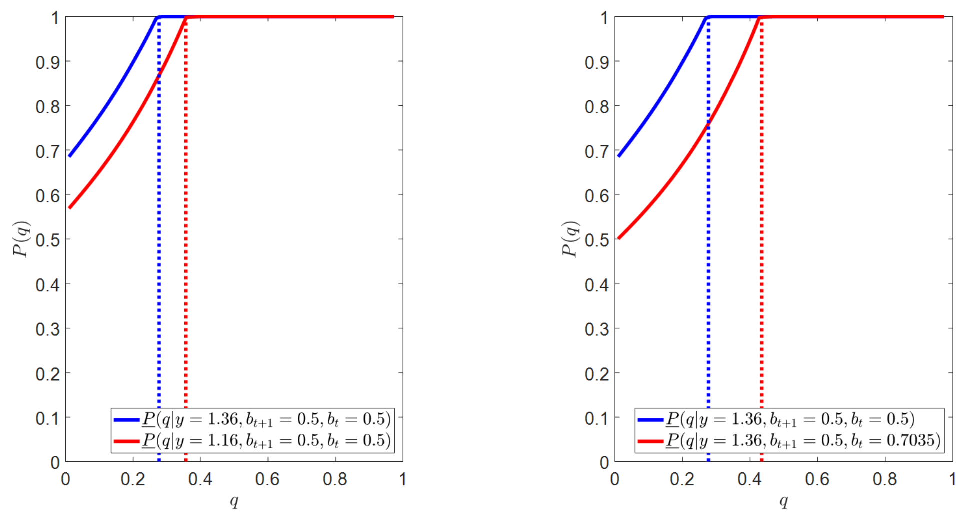

Figure 4: This figure plots $\underline{Q}\,(q)$ for different levels of output and for our main calibrated parameters. The left panel fixes $b_{t+1}$and b_t and shows the comparative statistics with respect to y_t. The right panel fixes y_t and shows the comparative statistics with respect to b_t.

Note that if the income realization is such that $\overline{V}^{nd}\,(b_t, y_t) = V^d\,(y_t)$ (i.e., under the best continuation equilibrium, the government is indifferent between defaulting or not, and still does not default), then $\underline{Q}\,(\hat{q}; b_t, y_t, b_{t+1}) = 0$ for any $\hat{q} < \underline{q}\,(b_t, y_t, b_{t+1}) = \overline{q}\,(y_t, b_{t+1})$. The idea is that for these income levels, only $q = \overline{q}\,(y_t, b_{t+1})$ is an equilibrium consistent price, and the only distribution that is equilibrium consistent places probability one on that price. Note also that $\underline{Q}$ is a cumulative distribution function for q: it is a non-increasing, right-continuous function with a range of $[0, 1]$; hence it implicitly defines a probability measure for debt prices.

Figure 4 presents the function for the maximum probability of prices, $\underline{Q}\,(\hat{q})$, for different states (b_t, y_t, b_{t+1}). In the left panel the two distributions differ on the income realization under which the government repaid its debt. Lets start with the blue line: the government repaid debt under an income realization of 1.36 (y_t), repaid 0.5 units of debt (b_t), and issued 0.5 units (b_{t+1}). $\underline{Q}\,(0)$ is approximately 0.7: the maximum probability of obtaining a price of zero is approximately 0.7. Any distribution where the probability is higher than 0.7, after $(b_t, y_t, b_{t+1}) = (0.50, 1.36, 0.50)$, is not equilibrium consistent because it would violate the IC constraint of the government. Second, note that as price q increases, $\underline{Q}\,(\hat{q})$ also increases: the government is willing to accept a higher probability of obtaining low prices (lower than $\hat{q}$), because these prices are not that low. Third, as we should expect, the function $\underline{Q}\,(\hat{q})$ reaches one at a price $\underline{q}(b_t, y_t, b_{t+1})\,|_{(b_t, y_t, b_{t+1})=(0.50,1.36,0.50)}$. Finally, note that the function $\underline{Q}(\hat{q})$ shifts if the government repays its debt under poor economic conditions (these conditions imply a lower spot utility); for example, $\underline{Q}(0)$ is approximately 0.55, if income is 1.16 instead of 1.36, which is what one would expect in order not to violate the incentive compatibility constraint. Finally, the right hand side of the panel shows the comparative statistics with respect to how much debt is repaid.

Bounding Expectations. One application that is of particular interest is bounding the moments of distributions across all equilibria. We start with expected values. The set of equilibrium consistent expected prices is just the set of possible $\int q dQ$ for some $Q \in \mathbb{ECD}(b_t, y_t, b_{t+1})$. Denote this set by $E(b_t, y_t, b_{t+1})$. We will show that this set can be easily characterized and is related to the prices we studied in the model without sunspots in Section 3. The following proposition shows that, in fact, the set of expected values is identical to the set of equilibrium consistent prices when there are no sunspots.

Proposition 5. *Suppose that history $h_m^{t+1} = (h^t, y_t, d_t, b_{t+1})$ is equilibrium consistent. Then the set of expected prices is equal to the set of prices without sunspots; i.e.,*

$$E(b_t, y_t, b_{t+1}) = \left[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})\right].$$

Moreover, if $b_{t+1} > 0$, then the minimum expected value is uniquely achieved at the Dirac distribution $\hat{Q}$ that assigns probability one to $q = \underline{q}(b_t, y_t, b_{t+1})$.

Proof. See Appendix B. □

The result follows from the concavity of the value function $\overline{v}(y_t, b_{t+1}, \hat{q})$ and the fact that $\underline{q}(\cdot)$ is the minimum price q for which $u(y_t - b_t + qb_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, q)$ is equal to $V^d(y_t)$. The equality at $q = \underline{q}(\cdot)$ follows from the strict monotonicity in q of equilibrium utility; given by $u(y_t - b_t + qb_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, q)$. If the inequality were to be strict, then we can find a lower equilibrium consistent price, which contradicts the definition of $\underline{q}(\cdot)$. Therefore, the integrand in the left hand side of 4.1 is larger than $V^d(y_t)$ only when $q \geq \underline{q}(b_t, y_t, b_{t+1})$. The concavity of $\overline{v}(y, b', q)$ and Jensen's inequality then imply that for any distribution $Q \in \mathbb{ECD}(b_t, y_t, b_{t+1}) : u\left(y_t - b_t + \mathbb{E}_Q(q) b_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, \mathbb{E}_Q(q)\right)$ has to be greater than or equal than $\int[u(y_t - b_t + qb_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, q)]dQ(q)$. The latter needs to be greater than or equal to $V^d(y_t)$ for Q to be an equilibrium consistent distribution, which explains why $\mathbb{E}_Q(q_t)$ is greater than $\underline{q}(b_t, y_t, b_{t+1})$.

Proposition 5 provides testable implications of equilibrium in the dynamic game that we are analyzing. These implications extend the restrictions derived in the work of Jovanovic (1989) and Pakes et al. (2015). The bounds that we just derived yield moment inequalities; in particular, that $\mathbb{E}_{q_t}\left[q_t \mid h_m^{t+1}\right] \in [\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})]$. Aided with these moment inequalities, one could in principle, perform estimation of the structural set of parameters as in Chernozhukov et al. (2007) and Galichon and Henry (2011).

Bounding Variances. Next, we characterize bounds over variances. The importance of this application comes not only from the fact that we can obtain dynamic implications from equilibria; we can also know, ex-ante, how much volatility the model can generate. Note that without any a priori knowledge this can be a daunting task. Which equilibrium will yield the highest variance? In the next proposition, we can pin down how much variance the model can generate, without trying every possible equilibrium. Take any $Q \in \mathbb{ECD}(h_m^{t+1})$ with $\mathbb{E}_Q(q_t) = \mu$. Denote by $S\left(h_m^{t+1}, \mu\right)$ the set of variances of these distributions.

Proposition 6. *Suppose that history* $h_m^{t+1} = \left(h^t, y_t, d_t, b_{t+1}\right)$ *is equilibrium consistent. Define* $q^* := \left[1 - \underline{Q}(0)\right] \times \overline{q}(y_t, b_{t+1})$. *If* $Q \in \mathcal{Q}$ *and* $\mathbb{E}_Q(q_t) = \mu$ *then* $S\left(h_m^{t+1}, \mu\right) = \left[0, \overline{\mathbb{V}ar}\left(h_m^{t+1}, \mu\right)\right]$ *where* $\overline{\mathbb{V}}\left(h_m^{t+1}, \mu\right)$ *is defined as:*

- *If* $\mu \geq q^*$, *then* $\overline{\mathbb{V}ar}\left(h_m^{t+1}, \mu\right) = \mu\left(\overline{q} - \mu\right)$.
- *If* $\underline{q}\left(b_t, y_t, b_{t+1}\right) \leq \mu < q^*$ *then* $\overline{\mathbb{V}ar}\left(h_m^{t+1}, \mu\right) = \mu\left(\overline{q} + q_\mu - \mu\right) - q_\mu \overline{q}$, *where* $\underline{Q}(q)$ *is defined in Proposition 4 and* q_μ *is the unique solution to the equation* $\underline{Q}\left(q_\mu\right) q_\mu + \left(1 - \underline{Q}\left(q_\mu\right)\right)\overline{q} = \mu$.

Proof. See Appendix B. □

The idea of the proof is as follows. We know that any price distribution with sunspots lies in the interval $[0, \overline{q}(y_t, b_{t+1})]$. We need to show that the maximum variance is achieved always with a binary distribution and absent any additional constraints, given the mean μ, the maximum variance is given by equation $\mu\left(\overline{q} - \mu\right)$. First, we show that the no default incentive constraint (4.1) is not binding if the expected prices are high enough; i.e., if $\mu \geq q^*$. Therefore, in this case, $\overline{\mathbb{V}ar}\left(h_m^{t+1}, \mu\right) = \mu\left(\overline{q} - \mu\right)$. When $\mu < q^*$, the incentive constraint for no-default starts to be binding. The maximum variance is still achieved by a binary distribution, but this binding constraint restricts how low the prices can be in the two values of the distribution. Thus, q_μ is defined as the value that solves $\Pr\left(q_\mu\right) q_\mu + \left(1 - \Pr\left(q_\mu\right)\right)\overline{q}$ equal to μ for some distribution Pr. Turns out that the distribution, Pr for which the incentive constraint (4.1) is binding is $\underline{Q}(\cdot)$. This is intuitive, because will make the probability of the low value as high as possible, maximizing the variance. It can also be shown that both q_μ and, hence, $\overline{\mathbb{V}ar}\left(h_m^{t+1}, \mu\right)$ are strictly increasing in μ.

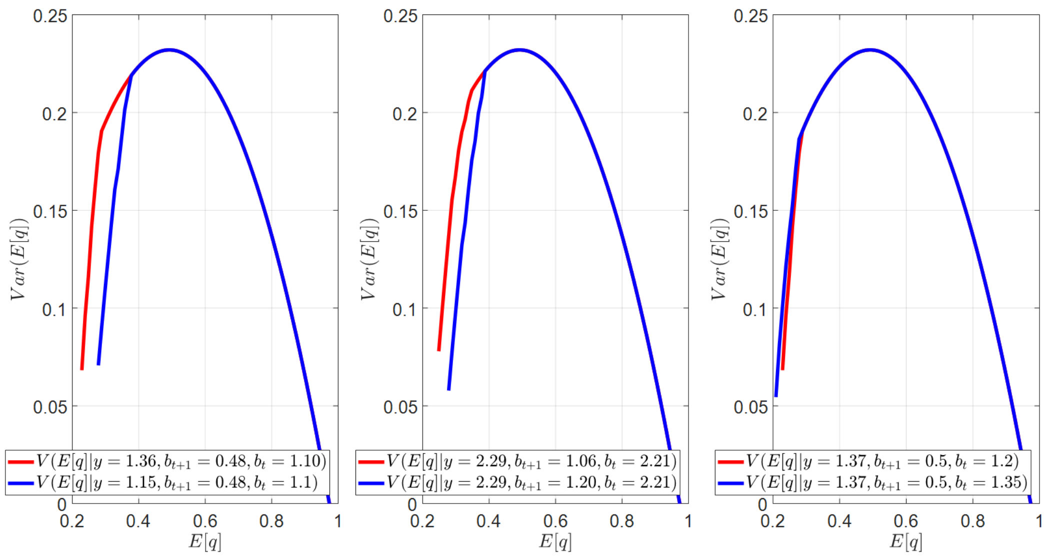

Figure 5: This figure plots $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right)$ for for different levels output and for our main calibrated parameters. The left panel fixes $b_{t+1}$and b_t and perform comparative statistics with respect to y_t. The middle panel fixes $y_t$and b_t and perform comparative statistics with respect to b_{t+1}.The right panel fixes y_t and perform comparative statistics with respect to b_t.

Figure 5 presents the bounds of the variances for the equilibrium consistent distributions given an expected value for prices. First, it is clear that in the three panels, the frontier of the mean and variances has kinks. All these kinks occur when the expected price is equal to q^*. Each one of the panels and each of the two cases in each panel are different because different values of (b_t, y_t, b_{t+1}). Second, note that in the left panel, both curves are the same up to the inflection point of the blue line. This result occurs because q^* is a function of (b_t, y_t, b_{t+1}), whichmarks the inflection point for each one of the curves. If the expectation of prices, $\mathbb{E}(q)$, is higher than the maximum of both q^*, then the functions are identical because in the left Panel y_t, b_{t+1} is the same for both frontiers. The blue line falls faster than the red line is because for the blue line the debt repayment is larger; thus, for a given mean the variance needs to be smaller. The middle panel also presents this same intuition but with respect to the issuance of new debt. Because more debt is issued in the red line, the spot utility for the government is relaxed. In addition, the right-hand side panel has the opposite intuition: when more debt is repaid, then the government tolerates lower variances. Finally, it is worth noting that for values of $\mathbb{E}(q)$ that are higher than q^*, the blue and red lines do not need to coincide. The reason why they coincide is because $\overline{q}(y_t, b_{t+1})$ is flat for both variables in the range of y_t, b_{t+1} in the plots.

Comparative Statics and Stochastic Dominance. We close this subsection by providing the comparative statistics over the set of distributions, $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$. This result parallels what we found in Corollary 1 in Subsection 3.3.

Corollary 2. *The set of equilibrium price distributions* $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ *is non-increasing (in a set order sense) with respect to* b_t *and if income is i.i.d, it is non-decreasing in* y_t*. Furthermore, suppose that* $Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ *and* Q' *is a probability distribution for equilibrium prices; i.e.* $Q' \in \Delta\left(\left[0, \overline{q}\left(y_t, b_{t+1}\right)\right]\right)$*. If* Q' *first order stochastically dominates (FOSD)* Q*, then* $Q' \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$*.*

Proof. See Appendix B. □

The intuition of the first part of these comparative statistics, again, stems from the revealed preference argument. If the government repaid a larger amount of debt, then the distribution of the prices that they would expect needs to shift towards higher prices. If the set does not change, then there will be a distribution that will be inconsistent with equilibrium because it will violate the promise-keeping constraint. For the second part, if Q' FOSD Q, then the proposition shows that once a distribution is consistent with equilibrium, any distribution that FOSD this distribution will be an equilibrium consistent distribution. Intuitively, higher prices lead to both higher consumption and higher continuation equilibrium values for the government since both are weakly increasing in the debt price q_t.[23]

5 A General Dynamic Policy Game

In this section we show that the main results that we proved in Section 3 and Section 4, Proposition 1 and Proposition 3, extend to a more general class of policy games and do not rely on the specific model studied in these sections. This should not be surprising. The main economic argument for Propositions 1 and 3 comes from a revealed preference: what the government leaves on the table provides bounds on the expectation it had regarding future play. These bounds, place restrictions over outcomes or over distributions.

[23] See that $\underline{Q}$ is, by its own definition, the infimum over all possible distributions in $\mathbb{ECD}$, since it gives the greatest cdf across all equilibrium consistent distributions. For every $Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ we have that Q FOSD $\underline{Q}$, and if Q' is some other lower bound, then Q' FOSD $\underline{Q}$. Moreover, $\underline{Q} \notin \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$. The distribution $\underline{Q}\left(\cdot\right)$ is the maximum lower bound (in the FOSD sense) of the set equilibrium consistent distributions; i.e. for every $Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ we have Q FOSD $\underline{Q}$, and if Q' is some other lower bound, then Q' FOSD $\underline{Q}$.

Therefore, in this section we do two things. First, we propose a general model of a dynamic policy game in the spirit of Stokey et al. (1989).[24] Second, we provide the analogs of Propositions 1 and 3 for this more general setup. In each one of these cases we show how to traduce these characterizations into restrictions over observables.

5.1 Setup

We will follow the notation in Stokey et al. (1989). There are two players: an infinitely long lived player (government) and short lived agents (market) that set expectations according to a particular rule. In each period t, agents play an extensive form stage game, with 5 sub periods $(t, \tau_i)_{i \in \{1,5\}}$. The payoff relevant states are an exogenous random shock y_t, and an endogenous state variable b_t. The timeline of the stage game follows:

- $\tau = \tau_1$: A publicly observable random variable $y_t \in Y \subseteq \mathbb{R}^l$ is realized, that follows a (controlled) Markov process: $y_t \sim f(y \mid y_{t-1}, b_t)$.[25]

- $\tau = \tau_2$: The long-lived player (government) chooses a control $d_t \in D \subseteq \mathbb{R}^d$ and a next period state variable $b_{t+1} \in B \subset \mathbb{R}^b$ (where both D and B are compact sets) that are jointly feasible, given (b_t, y_t). We say that (d_t, b_{t+1}) is feasible if $(d_t, b_{t+1}) \in \Gamma(b_t, y_t)$, where $\Gamma : B \times Y \times Q \rightrightarrows D \times B$ is a non-empty, compact valued , continuous correspondence.

- $\tau = \tau_3$: A sunspot variable ζ_t is realized and distributed according to $\zeta_t \sim U[0,1]$.

- $\tau = \tau_4$: The agents determine their expectations about future play. This process is modeled in reduced form, with the market choosing $q_t \in \mathbb{R}^k$ to satisfy:

$$q_t = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \delta^{s-t} T(b_{s+1}, y_{s+1}, d_{s+1}, b_{s+2}) \right\}$$

[24] To keep notation simple and the exposition more concrete, we will focus on games in which the short run players form an expectation regarding next period policy. There is a large class of models that share this timing. For sovereign debt, one class follows Eaton and Gersovitz (1981). For monetary policy, one class is the New Keynesian model as in Benigno and Woodford (2003). There are policy games that focus on alternative timings, though. In particular, there is a class of games in which the decision of the long-lived player and the short-lived players occurs sequentially, but in the same period. This timing has been used mainly for monetary policy (for example, in the seminal contribution of Barro and Gordon, 1983, but see also, for example, Obstfeld et al., 1996), and capital taxation (see for example Phelan, 2006 and Chari and Kehoe, 1990). Our results can be extended to incorporate these alternative timings.

[25] Sometimes, we say that y *includes a sunspot* if $\exists \{y_t^*, z_t\}$ such that (1) $y_t^* \perp z_t$ for all t, (2) y_t^* is a controlled Markov process; i.e. $y_t^* \sim g(y_t^* \mid y_{t-1}^*, b_t)$ and (3) $z_t \sim_{i.i.d}$ Uniform $[0,1]$.

where $\delta \in (0,1)$ and $T : B \times Y \times D \times B \to \mathbb{R}^k$ is a continuous, bounded function. The expectation is taken over future shocks $\{y_{t+s}\}_{s=1}^{\infty}$ knowing the strategy profile of the long lived player.

- $\tau = \tau_5$: the payoffs for the long lived player are realized and given by a continuous utility function $u(b_t, y_t, d_t, b_{t+1}, q_t)$. Lifetime utility is then given by

$$V_0 := \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(b_t, y_t, d_t, b_{t+1}, q_t) \right\},$$

where $\beta \in (0,1)$.

Example 1. This example is exactly the one studied in Section 3 and Section 4. y_t is national income, $b_t \geq 0$ is the outstanding public debt to be repaid, $d_t \in \{0,1\}$ is the default decision and $q_t = \mathbb{E}_t \left[\frac{1-d_{t+1}}{1+r} \right]$ is the risk neutral price set by lenders in equilibrium. Flow utility is given by $u(b_t, y_t, d_t, b_{t+1}, q_t) = (1-d_t) u(y_t - b_t + q_t b_{t+1}) + d_t u(y_t)$, assuming that when the government defaults on its debt, it gets to consume its income and is banned forever from international financial markets. Note that the feasibility correspondence is given by $\Gamma(y_t, b_t, q_t) = y_t - b_t + q_t b_{t+1} \geq 0$.[26]

Example 2. Our framework also incorporates New Keynesian (NK) models of monetary policy with no endogenous state; see for example Benigno and Woodford (2003) and more recently Waki et al. (2015). In the case of the NK model the control is $d_t = \pi_t$ where π_t is inflation. Agents set inflation expectations to match future inflation, as $q_t := \pi_t^e = \mathbb{E}_t(\pi_{t+1})$. Inflation and output are related according to a forward looking Phillips curve $g_t = \pi_t - \beta \pi_t^e + \epsilon_t$, where g_t is the output gap and ϵ_t is a supply shock. In addition, let π_t^* be a random variable that gives the optimal natural level of inflation (absent an inflation gap). The random shocks are then $y_t = (\epsilon_t, \pi_t^*)$, and the government is assumed to minimize the loss function:

$$\mathcal{L}(\pi, \pi^e, \epsilon_t, \pi_t^*) = \frac{1}{2} g_t^2 + \frac{1}{2} \chi (\pi_t - \pi_t^*)^2 = \frac{1}{2} (\pi_t - \beta \pi_t^e + \epsilon_t)^2 + \frac{1}{2} \chi (\pi_t - \pi_t^*)^2.$$

In this example, the feasibility constraint represents the fact that π_t needs to be bounded.

Histories, Equilibrium and Equilibrium Consistency. The notation in this section follows the one used in Sections 2, 3, and 4. Recall that a *history* is a vector $h^t = (h_0, h_1, ..., h_{t-1})$,

[26]As we commented in Section 2, because the market chooses after the government it can be the case that this constraint is ex-post "violated". In that case, the government has a technology available to generate resources such that the budget constraint holds; in this case the government obtains utility of $-\infty$.

where $h_t = (y_t, d_t, b_{t+1}, q_t)$ is the description of the outcome of the stage game at time t. A partial history is an initial history h^t concatenated with some subset of the stage game at period t. The set of all partial histories (initial and partial) is denoted by $\mathcal{H}$, and $\mathcal{H}_g \subset \mathcal{H}$ represent the histories where the government has to choose (d_t, b_{t+1}); i.e., $h^t_g = \left(h^t, y_t\right)$. Likewise, $\mathcal{H}_m \subset \mathcal{H}$ is the set of partial histories where the expectation setters (or "market") ; i.e., $h^{t+1}_m = \left(h^t, y_t, d_t, b_{t+1}\right)$. A *strategy* for the *government* is a function $\sigma_g\left(h^t, y_t\right) = (d_t, b_{t+1})$ for all histories, and a strategy for the *market* is a pricing function $q_m\left(h^t, y_t, d_t, b_{t+1}, \zeta_t\right) \in \mathbb{R}_k$. The payoff for the government of a particular (feasible) strategy σ_g, σ_m , after a particular history h^t, y_t is given by:

$$V\left(\sigma \mid h^t, y_t\right) = \mathbb{E}_t\left\{\sum_{t=s}^{\infty} \beta^{t-s} u\left(b_t^{\sigma_g}, y_t, d_t^{\sigma_g}, b_{t+1}^{\sigma_g}, q_t^{\sigma_m}\right)\right\}.$$

A strategy profile $\sigma = \left(\sigma_g, q_m\right)$ is a *Subgame Perfect Equilibrium* (SPE) of the game if:

a. $V\left(\sigma \mid h^t, y_t\right) \geq V\left(\sigma'_g, q_m \mid h^t, y_t\right)$ for all $\left(h^t, y_t\right), \sigma'_g \in \Sigma_g$;

b. $q_m\left(h^t, y_t, d_t, b_{t+1}, \zeta_t\right) = \mathbb{E}_t\left\{\sum_{s=t}^{\infty} \delta^{s-t} T\left(b_{s+1}, y_{s+1}, d_{s+1}, b_{s+2}\right)\right\}$ where the policies $(b_{s+1}$ $d_{s+1}, b_{s+2})$ are generated by σ.

We denote it by $\sigma \in \Sigma^*$. The methodology we developed in Sections 3, and 4, derived statistical predictions for the data generated by the set of subgame perfect equilibria. We focused in a particular dynamic policy game that followed Eaton and Gersovitz (1981). In this section we follow similar steps for the general model that we just described. Given a SPE profile $\sigma = \left(\sigma_g, q_m\right)$, we define its *equilibrium path* $\pi = x(\sigma)$ as a sequence of measurable functions $\pi = \left(d_t\left(y^t\right), b_{t+1}\left(y^t\right), q_t\left(y^t\right)\right)_{t \in \mathbb{N}}$ that are generated by following the profile σ. A history h is *equilibrium consistent* if and only if is on some equilibrium path $x = x(\sigma)$, for subgame prefect equilibrium $\sigma \in \Sigma^*$.

What follows? First, in subsection 5.2, we characterize the worst equilibrium payoff and the best possible continuation after a realization of the expectation of the public. Recall that the best continuation value function played a central role in the characterization of equilibrium consistent distributions in Proposition 3. As we explained after discussing Proposition 3, this object is also useful for the characterization of equilibrium consistent outcomes without sunspots. Second, in subsection 5.3, paralleling what we did in Section 3 we will characterize equilibrium consistent outcomes, for the model when there are no sunspots. The main result is Proposition 7. Finally, paralleling what we did in Section 4, we will characterize equilibrium consistent distributions over outcomes. The main result is Proposition 8 which is an extended version of Proposition 3.

5.2 Equilibrium and Continuation Values

As we did in Section 3, it will be useful to define the best ex-post continuation payoff. Also, we define the set of equilibrium payoffs and and the worst equilibrium payoff. We start with the set of *equilibrium payoffs*. Formally, denote as $\mathcal{E}(y_-, b)$ and $\mathcal{E}^s(y_-, b)$ the set of equilibrium payoff in the model without and with sunspots, respectively. Formally, $\mathcal{E}(y_-, b)$ is defined as:

$$\mathcal{E}(y_-, b) = \Big\{ (q, v) \in \mathbb{R}^k \times \mathbb{R} : \exists \sigma \in \Sigma^*(y_-, b) \text{ with} \\ \left. \begin{array}{c} v = V(\sigma \mid h_0 = (y_-, b)) \\ q = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \delta^t T(b_{t+1}, y_{t+1}, d_{t+1}, b_{t+2}) \mid y_-, b \right\} \end{array} \right\}$$

and let $\mathcal{Q}(y_-, b) \subseteq \mathbb{R}^k$ be its projection over q. We can characterize $\mathcal{E}(y_-, b)$ using the concept of self-generation and enforceability in Abreu (1988); Abreu et al. (1990) and Atkeson (1991). We can show that if y is non-atomic and u is concave in q (for example, risk aversion of the long lived player), then $\mathcal{E}(y_-, b)$ is compact and convex valued. This is satisfied by both examples discussed above. Furthermore, if $\mathcal{E}(y_-, b)$ is compact and convex valued, then $\mathcal{E}^s(y_-, b) = \mathcal{E}(y_-, b)$.[27] For a simpler exposition we focus on this case.[28]

We continue with the *best value function* and the *max-min value*. The best value function gives the maximum equilibrium value for the long lived player, if $q_t = q_-$ is realized; i.e.,

$$\overline{v}(y_-, b, q_-) = \max_{v \in \mathbb{R}} v$$

$$\text{s.t. } (q_-, v) \in \mathcal{E}(y_-, b).$$

By following steps that are similar to the ones used in the Appendix, Section D, we can also show that if $\mathcal{E}(y_-, b)$ is convex valued and $u(\cdot)$ is concave in q, then $\overline{v}(y_-, b, q_-)$ is also concave in q. The *max-min value* is the worst possible value that the long lived player

[27] In repeated games, it is usually the case that for the set of equilibrium payoffs with and without sunspots to be equal, we only need to know that the set without sunspots is convex. However, our case is different because the continuation value of one of short lived players enters non-linearly in the utility function of the long lived players, and therefore in the IC constraint of the other one. Which explains why, in addition to convexity, we need concavity of the spot utility function with respect to q. This was satisfied in the model of sovereign debt in our first sections.

[28] Why is the exposition simpler? When this is not the case, all the propositions in this section remain valid, but we need to define the functions $\overline{v}$, the best ex-post continuation payoff and $\underline{U}$, the worst equilibrium payoff, over the correspondence $\mathcal{E}^s(y_-, b)$ instead. These two functions are defined below $\overline{v}^s(y, b, q) = \max\{v : (q, v) \in \mathcal{E}^s(y, b)\}$ and $\underline{U}^s(y, b) := \max_{(d, b') \in \Gamma(b, y)} \min_{(q, v) \in \mathcal{E}^s(y, b)} u(b, y, d, b', q) + \beta v$.

can obtain in any SPE , going forward. Formally,

$$\underline{U}(y,b) := \max_{(d,b')\in\Gamma(b,y)} \left\{ \min_{(q,v)\in\mathcal{E}(y,b')} u\left(b,y,d,b',q\right) + \beta v \right\}.$$

How this is related to what we did in Sections 3, and 4? In the sovereign debt model, $\underline{U}(y,b) = V^d(y)$, denotes the autarky value. We show this in the Appendix C.[29]

5.3 Equilibrium Consistency

Let us start by analyzing the case *without a sunspot* after the decision of the government. Aided with the best ex-post continuation $\overline{v}(y,b',q)$ and the max-min value for the government, the main result of this section is to characterize which period t outcomes $h_t = (d_t, b_{t+1}, q_t)$ are equilibrium consistent, after an equilibrium consistent history h^t. These outcomes are denoted by $\mathbb{ECO}\left(h^t\right)$. We then apply Proposition 7 to obtain predictions over q_t across all equilibria as we did for the model of sovereign debt.

Proposition 7. *Suppose that h^{t+1} is an equilibrium consistent history. Then, an outcome $h_{t+1} = (d_t, b_{t+1}, q_t)$ is equilibrium consistent if and only if: (a) q_t is an equilibrium prices; i.e. $q_t \in \mathcal{Q}(y_t, b_{t+1})$; and (b) incentive compatibility for the long lived player:*

$$u(b_t, y_t, d_t, b_{t+1}, q_t) + \beta\overline{v}(y_t, b_{t+1}, q_t) \geq \underline{U}(y_t, b_t).$$

The proof of Proposition 7 follows closely the steps of the proof of Proposition (3). We briefly discuss the argument in the Appendix Section E. Proposition 7 identifies the necessary and sufficient conditions for an outcome (d_t, b_{t+1}, q_t) to be equilibrium consistent after an equilibrium consistent history. There are a couple of points that are worth noting. First, the condition that $q_t \in \mathcal{Q}(y_t, b_{t+1})$ just states that q_t needs to be an equilibrium price or expectation. In the model of sovereign debt, it stated that $q_t \in [0, \overline{q}(y_t, b_{t+1})]$; i.e., that the price was between zero and the price of the best equilibrium. Clearly, if $q_t \notin \mathcal{Q}(y_t, b_{t+1})$ then it cannot be part of a continuation equilibrium, so q_t would not be equilibrium consistent.

Second, the IC constraint is replaced by only one equation. Necessity is intuitive. If for any $y_t \in \mathbb{Y}$, the condition is violated, then the policies d_t, b_{t+1} could not be implemented by promising the best continuation equilibrium if they follow them, and the worst continuation if they do not. Thus, they cannot be implemented. The sufficiency of this condition

[29]There are several papers that develop the techniques to solve for the set of equilibrium payoffs following the seminal contribution of Judd et al. (2003). Following Waki et al. (2015), it can be shows that $\overline{v}(y_-, b, q_-)$ can be expressed as the unique fixed point of a contraction mapping, given $\underline{U}(y,b)$.

again stems from the fact that y_t is non-atomic, and hence, any particular realization of y_t has no marginal effect on expected lifetime utilities from the previous periods; i.e. the promise keeping constraints can always be satisfied if we change the realization of the continuation value on a single y_t.[30]

How can we use the previous Proposition to obtain robust *predictions on prices*? The second condition in Proposition 7 defines, for a given y_t and the long lived player's choice (d_t, b_{t+1}), a set of equilibrium consistent prices:

$$\mathbb{ECO}\left(b_t, y_t, d_{t,}, b_{t+1}\right) := \left\{q_t \in \mathcal{Q}\left(y_t, b_{t+1}\right) : u\left(b_t, y_t, d_t, b_{t+1}, q_t\right) + \beta \overline{v}\left(y_t, b_{t+1}, q_t\right) \geq \underline{U}\left(y_t, b_t\right)\right\}. \tag{5.1}$$

If $\overline{v}\left(y_t, b_{t+1}, q_t\right)$ is concave in q_t, which happens if $\mathcal{E}$ is convex valued and u is concave in q, then the set of equilibrium consistent prices, that we denote as $\mathbb{Q}\left(b_t, y_t, d_t, b_{t+1}\right)$ will be convex. In the case of $k = 1$, this implies that $\mathbb{Q}$ is a compact interval; $\mathbb{Q}\left(b_t, y_t, d_t, b_{t+1}\right) = \left[\underline{q}\left(b_t, y_t, d_t, b_{t+1}\right), \overline{q}\left(b_t, y_t, d_t, b_{t+1}\right)\right]$ as in the sovereign debt model. Again, as in the previous sections, the set $\mathbb{Q}\left(b_t, y_t, d_t, b_{t+1}\right)$ defines the restrictions over observables of the assumption of equilibrium.

We now go back to the case in which there is a *sunspot (public correlating device)* that is realized in $\tau = 4$. We present a generalization of the main result presented in Section 4, Proposition 7, for the general model that we just introduced. We will assume that $\mathcal{E}\left(y_-, b\right)$ is convex valued and u is concave in q.

Proposition 8. *Suppose that h^{t+1} is an equilibrium consistent history. Then, Q_t is an equilibrium consistent distribution if and only if: (a) $Q_t \in \Delta\left[\mathcal{Q}\left(y_t, b_{t+1}\right)\right]$;* (b) *incentive compatibility for long lived player:*

$$\int_{\hat{q} \in \mathcal{Q}\left(y_t, b_{t+1}\right)} \left[u\left(b_t, y_t, d_t, b_{t+1}, \hat{q}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right] dQ_t\left(\hat{q}\right) \geq \underline{U}\left(y_t, b_t\right).$$

As for the previous proposition presented in this section the proof of Proposition 7 follows closely the steps of the proof of Proposition 3. We discuss the argument in the Online Appendix Section E. Proposition 8 generalizes Proposition 7 for the case with sunspots, when $\mathcal{E}$ is convex valued and u is concave in q. Again, the first requirement, $Q_t \in \Delta\left[\mathcal{Q}\left(y_t, b_{t+1}\right)\right]$ is asking for a distribution to be a probability distribution over equilibrium prices. As in the case without sunspots, and more importantly, as in Section 4, we can use the results in Proposition 8 to obtain observable implications over prices. For example, we can again obtain bounds over expectations. Define now a set of equilibrium

[30] One final comments. Note that we can characterize all equilibrium consistent histories recursively: start with the null history $h^0 = \left(y_-, b_0\right)$ (the starting state) and, h^{t+1} is equilibrium consistent if and only if h^t is equilibrium consistent and $h_t = \left(y_t, d_t, b_{t+1}, q_t\right)$ satisfies conditions (1) and (2) of Proposition 7.

consistent price distributions $\mathbb{ECD}\left(b_t, y_t, d_t, b_{t+1}\right)$. Because the IC for the government is a linear inequality on measures Q_t, under the assumptions of Proposition 8 the function $g\left(\hat{q} \mid h_t\right) := u\left(b_t, y_t, d_t, b_{t+1}, \hat{q}\right) + \beta\overline{v}\left(y_t, b_{t+1}, \hat{q}\right) - \underline{U}\left(y_t, b_t\right)$ is concave in $\hat{q}$ as well. Therefore, as in the sovereign debt model, we have that the set of expected prices $E\left(b_t, y_t, b_{t+1}\right)$ that is defined as the values $\int \hat{q} dP\left(\hat{q}\right)$ such that $Q \in \mathbb{ECD}\left(b_t, y_t, d_t, b_{t+1}\right)$. In this case, we can show that is equal to the set of equilibrium consistent prices without sunspots; i.e. $E\left(b_t, y_t, d_t, b_{t+1}\right) = \mathbb{Q}\left(b_t, y_t, d_t, b_{t+1}\right)$.

6 Conclusion

Dynamic policy games have been extensively studied in macroeconomic theory to increase our understanding of how lack of commitment restricts the outcomes that a government can achieve. One of the challenges in studying dynamic policy games is equilibrium multiplicity. Our paper acknowledges and embraces equilibrium multiplicity. For this reason, we focus on obtaining *robust predictions*. These are predictions that hold across all equilibria, or in the language of Bergemann and Morris (2017) across every possible information structure.

We obtain robust predictions by characterizing what we termed as equilibrium consistent outcomes. As we have discussed in the text, the basis of our predictions is a revealed preference argument, which was also exploited to obtain the testable implications of equilibria in Jovanovic (1989) and Pakes et al. (2015). The idea of the revealed preference argument is that by taking a particular action, the government obtained some utility; and by doing so may have left something else on the table. What the government left on the table, places bounds on what it can receive in the future. As we discuss in the text, equilibrium consistency is a general principle. Even though we focus on a model of sovereign debt that follows Eaton and Gersovitz (1981), as we show in the last section of the paper, our results can be generalized to other dynamic policy games.

We think that the predictions we obtain, in particular, the bounds on moments across distributions, provide testable implications of a model that are not sensitive to a particular equilibrium selection mechanism, and in addition, can be the basis of estimation procedures robust to equilibrium selection. These estimation procedures, can be extensions of the ones in an extensive literature in industrial organization and game theory (for example Berry, 1992, Bajari et al., 2007, Aguirregabiria and Mira, 2007) and Econometrics (for instance Chernozhukov et al., 2007 and Galichon and Henry, 2011) that recovers structural parameters of interest using moment conditions. These are topics for further research.

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Appendix to "Robust Predictions in Dynamic Policy Games"

Juan Passadore and Juan Xandri

A Proofs Main Results

Proof. Proposition 1. *Step 1: Necessity.* ($\Longrightarrow$). If $(d(\cdot), b'(\cdot))$ is SPE-consistent, there exists an SPE profile $\hat{\sigma}$ such that $h^t \in \mathcal{H}(\hat{\sigma})$ and

$$d(y_t) = d_t^{\hat{\sigma}}\left(h^t, y_t\right) \text{ and } b'(y) = b_{t+1}^{\hat{\sigma}}\left(h^t, y_t, d=0\right).$$

That is, there exists a SPE that can generate the history h_-^t, which specifies the contingent policy $d(\cdot), b'(\cdot)$ in period t, and satisfies conditions (3.4) to (3.6). Because $\hat{\sigma}$ is an SPE, using the results of Abreu et al. (1990) we know that if $d(y) = 0$ at $h^t = \left(h_-^t, q_{t-1}\right)$ then

$$u\left(y_t - b_t + b'(y_t)\, q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right)\right) + \beta V\left(\hat{\sigma} \mid h^{t+1}\right) \geq V^d(y_t). \tag{A.1}$$

According to the definition of best continuation values and prices.

$$V\left(\hat{\sigma} \mid h^{t+1}\right) \leq \overline{\mathbb{V}}\left(y_t, b'(y_t)\right) \text{ and } q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right) \leq \overline{q}\left(y_t, b'(y_t)\right). \tag{A.2}$$

Because $b'(y_t) \geq 0$ (the no savings assumption) and $u(\cdot)$ is strictly increasing, we can insert (A.2) into (A.1) to conclude that

$$u\left(y_t - b + b'(y_t)\,\overline{q}\left(y_t, b'(y_t)\right)\right) + \beta\overline{\mathbb{V}}\left(y_t, b'(y_t)\right) \geq$$

$$u\left(y - b_t + b'(y_t)\, q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right)\right) + \beta V\left(\hat{\sigma} \mid h^{t+1}\right)$$

which proves condition (3.5). Further, since $\hat{\sigma}$ generated the observed history, the past prices must be consistent with policy $(d(\cdot), b'(\cdot))$. Formally:

$$\begin{aligned} q_{t-1} &= q_m^{\hat{\sigma}}\left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right) \\ &= \frac{\mathbb{E}_{y_t|y_{t-1}}\left(1 - d^{\hat{\sigma}}\left(h^t, y_t\right)\right)}{1+r} \\ &= \frac{\mathbb{E}_{y_t|y_{t-1}}\left(1 - d^{\hat{\sigma}}\left(y_t\right)\right)}{1+r} \end{aligned}$$

which also proves (3.4). Using the usual promise keeping accounting, condition (3.6), is the same as condition (3.5) but at $t-1$. Formally, if $\hat{\sigma}$ is SPE and $h^t \in \mathcal{H}(\hat{\sigma})$ then the government's default and bond issue decision at $t-1$ was optimal given the observed expected prices

$$u\left(y_{t-1}-b_{t-1}+b_t q_{t-1}\right)+\beta V\left(\hat{\sigma} \mid h^t\right) \geq u\left(y_{t-1}\right)+\beta \mathbb{E}_{y_t \mid y_{t-1}} V^d(y_t).$$

Using the recursive formulation of $V(\cdot)$, we obtain the following inequality:

$$\begin{aligned}
V\left(\hat{\sigma} \mid h^t\right) &= \mathbb{E}_{y_t \mid y_{t-1}}\left[(1-d(y_t))\left[u(y_t-b_t+b'(y_t)q_m^{\hat{\sigma}}(h^t,y_t,d_t=0,b'(y_t)))+V\left(\hat{\sigma} \mid h^{t+1}\right)\right]\right] \\
&\quad+\mathbb{E}_{y_t \mid y_{t-1}}\left[d(y_t)\left[u\left(y_t\right)+\beta \mathbb{E}_{y_t \mid y_{t-1}} V^d(y_t)\right]\right] \\
&\leq \mathbb{E}_{y_t \mid y_{t-1}}(1-d(y_t))\left[u\left(y_t-b_t+b'(y_t)\overline{q}(b'(y_t))\right)+\overline{\mathbb{V}}(y_t,b'(y_t))\right] \\
&\quad+\mathbb{E}_{y_t \mid y_{t-1}} d(y_t)\left[u\left(y_t\right)+\beta \mathbb{E}_{y_t \mid y_{t-1}} V^d(y_t)\right].
\end{aligned}$$

According to the previous two inequalities and the law of iterated expectations, condition (3.6) follows.

Step 2: Sufficiency. ($\Longleftarrow$). We need to construct a strategy profile σ that is a SPE such that $h_m^t \in \mathcal{H}(\sigma)$ and $d(\cdot)=d_t^{\sigma}\left(h^t,\cdot\right)$ and $b'(\cdot)=b_{t+1}^{\sigma}\left(h^t,\cdot\right)$. Given that $h_m^t \in \mathcal{H}(\Sigma^*)$, we know there exists a SPE profile $\hat{\sigma}=\left(\hat{\sigma}_g,\hat{q}_m\right)$ that generates h_m^t. Let $\overline{\sigma}(b,y)$ be the best continuation SPE (associated with the best price $\overline{q}(\cdot)$) when $y_t=y$ and $b_{t+1}=b$. Let σ^{aut} be the strategy profile for autarky (associated with $q_m=0$ for all continuation histories). In addition, let $h^{t+1}(y_t)=\left(h^t,y_t,d(y_t),b'(y_t),\overline{q}(y_t,b'(y_t))\right)$ be the continuation history at $y_t=y$ and the policy $(d(\cdot),b'(\cdot))$ if the government faces the best possible prices. We define the histories that precede h^t and are not equal to h^t as $(h^s,y_s) \prec h^t$. That is, if we truncate h^t to period s, we obtain h^s. We denote $(h^s,y_s) \not\prec h^t$ as the histories that do not precede h^t. The symbol $\preceq$ denotes histories that precede and can be equal. We construct the following strategy profile $\sigma=\left(\sigma_g,q_m\right)$:

$$\sigma_g\left(h^s,y_s\right)=\begin{cases}
\hat{\sigma}_g\left(h^s,y_s\right) & \text{for all } (h^s,y_s) \prec h^t \\
\sigma^{aut}\left(y_s\right) & \text{for all } s<t \text{ and } (h^s,y_s) \not\prec h^t \\
d_t\left(h^t,y_t\right)=d\left(y_t\right) \text{ and } b_{t+1}\left(h^t,y_t\right)=b'\left(y_t\right) & \text{for } \left(h^t,y_t\right) \text{ for all } y_t \\
\overline{\sigma}_g\left(b_{s+1},y_s\right)\left(h^s,y_s\right) & \text{for all } h^s \succeq h^{t+1}\left(y_t\right) \\
\sigma^{aut}\left(y_s\right) & \text{for all } s>t,\ h^s \not\succ h^{t+1}\left(y_t\right)
\end{cases}$$

and

$$q_m\left(h^s, y_s, d_s, b_{s+1}\right) = \begin{cases} \hat{q}_m\left(h^s, y_s, d_s, b_{s+1}\right) & \text{for all } \left(h^s, y_s\right) \prec h^t \\ 0 & \text{for all } s < t \text{ and } \left(h^s, y_s\right) \not\prec h^t \\ \overline{q}\left(y_s, b'\left(y_s\right)\right) & \text{for all } h^s \succeq \left(h^t, y_t, d\left(y_t\right), b'\left(y_t\right)\right) \\ 0 & \text{for all } h^s /\!\succ \left(h^t, y_t, d\left(y_t\right), b'\left(y_t\right)\right). \end{cases}$$

By construction $h_m^t \in \mathcal{H}(\sigma)$. This occurs because $\sigma = \hat{\sigma}_g$ for histories $\left(h^s, y_s\right) \preceq h^t$. In addition, the strategy σ prescribes the policy $\left(d\left(\cdot\right), b'\left(\cdot\right)\right)$, which is on the equilibrium path. Next, we need to show that at h^t, this strategy profile is indeed an SPE. To do this, we will use the one deviation principle. Note that for all histories with $s > t$ the continuation profile is an SPE (by construction); this process prescribes the best continuation equilibrium, which is an SPE by definition. Next, we need to show that at h^t, this profile is indeed an equilibrium. The fact that an equilibrium exists is due to the second constraint, which is the IC constraint

$$\begin{aligned}(1-d(y_t))\left[u(y_t - b_t + \overline{q}(y_t, b_{t+1}((y_t)))b_{t+1}(y_t)) + \beta\overline{\mathbb{V}}(y_t, b_{t+1}((y_t)))\right] + ..\\ ... + d(y_t)V^d(y_t) \geq V^d(y_t).\end{aligned}$$

Note also that the default policy at $t-1$ was consistent with σ (and is an equilibrium) and that q_{t-1} is consistent with the policy $\left(d\left(\cdot\right), b'\left(\cdot\right)\right)$. The promise-keeping constraint (3.6) translates into the exact IC constraint for profile σ, showing that the default decision at $t-1$ was indeed optimal given profile σ. The "price keeping" (3.4) constraint also implies that q_{t-1} was consistent with policy $\left(d\left(\cdot\right), b'\left(\cdot\right)\right)$. The final step for proving sufficiency is showing that, $s < t-1$ (that is $h^s \prec h^t$). Note that because y is absolutely continuous, the particular y that is realized has zero probability. Therefore, the expected value of this new strategy is the same

$$V(\hat{\sigma} \mid h^s) = V(\sigma \mid h^t)$$

for all $h^s \prec h^t$ with $s < t-1$; the probability of the realization of h^t, is zero. Altogether,this implies that σ is indeed an SPE and generates history h_m^t on the equilibrium path, proving the desired result. □

Proof. Proposition 2. *Step 1: Rewrite program* $\mathbb{ECO}$ *program.* By Proposition 1 and $\underline{q}$ is defined as:

$$\underline{q}\left(h_m^{t-1}\right) = \min_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q}$$

subject to

$$(\hat{q}, d_t(\cdot), b_{t+1}(\cdot)) \in \mathbb{ECO}(b_{t-1}, y_{t-1}, b_t).$$

This can be rewritten as:

$$\underline{q}(b, y, b') = \min_{q, d(\cdot) \in \{0,1\}^Y, b''(\cdot)} q$$

subject to

$$\frac{\mathbb{E}_{y'|y}[1 - d(y')]}{1+r} = q \tag{A.3}$$

$$(1 - d(y'))\left(\overline{V}^{nd}(b', y', b''(y')) - V^d(y')\right) \geq 0 \tag{A.4}$$

$$\beta \mathbb{E}_{y'|y}\left[d(y') V^d(y') + (1 - d(y')) \overline{V}^{nd}(b', y', b''(y'))\right] - .. \\ .. - \beta \mathbb{E}_{y'|y} V^d(y') \geq u(y) - u\left(y - b + b'q\right). \tag{A.5}$$

First, note that we can relax the constraints (A.4) and (A.5) by choosing:

$$b''(y') = \underset{\hat{b} \geq 0}{\operatorname{argmax}}\, \overline{V}^{nd}\left(b', y', \hat{b}\right).$$

Second, we define the set $R(b') = \left\{y' \in Y : \overline{V}^{nd}(b', y') \geq V^d(y')\right\}$ to be the set of income levels for which the government does not default, under the best continuation equilibrium. Note that if $y' \notin R(b')$, then the no default decision is not equilibrium feasible for any continuation equilibrium (this stems from the fact that (A.4) is a necessary condition for no default). The minimization program, that we now call program $P_{\underline{q}}$, can now be written as:

$$\underline{q}(b, y, b') = \min_{q, d(\cdot) \in \{0,1\}^Y} q \tag{A.6}$$

subject to

$$
\begin{aligned}
\frac{\mathbb{E}_{y'|y}\left[1-d\left(y'\right)\right]}{1+r} &= q \\
\left(1-d\left(y'\right)\right)\left[\overline{V}^{nd}\left(b',y'\right)-V^{d}\left(y'\right)\right] &\geq 0 \text{ for all } y' \in R\left(b'\right) \\
d\left(y'\right) &= 1 \text{ for all } y' \notin R\left(b'\right) \\
\beta\mathbb{E}_{y'|y}\left[d\left(y'\right)V^{d}\left(y'\right)+\left(1-d\left(y'\right)\right)\overline{V}^{nd}\left(b',y',b''\left(y'\right)\right)\right]-.. & \\
..-\beta\mathbb{E}_{y'|y}V^{d}(y') &\geq u\left(y\right)-u\left(y-b+b'q\right).
\end{aligned}
\tag{A.7}
$$

Step 2: Show that program $P_{\underline{q}}$ has a non-empty feasible set. As a preliminary step, we need to show that this problem has a non-empty feasible set. To accomplish this, we choose the default rule that makes all constraints less binding: i.e. $d\left(y'\right)=0 \iff \overline{V}^{nd}\left(b',y'\right) \geq V^{d}\left(y'\right)$. This rule corresponds to the best equilibrium policy. If this policy is not feasible, then the feasible set is empty. Under this default policy and at the best equilibrium, price q is equal to the best equilibrium price $q=\overline{q}\left(y,b'\right)$. The feasible set is non-empty if and only if

$$\overline{V}^{nd}\left(b,y,b'\right) \geq V^{d}\left(y\right).$$

Step 3: Rewrite the promise keeping constraint (A.7). Note that

$$V^{d}(y')=\left[d\left(y'\right)V^{d}\left(y'\right)+(1-d\left(y'\right))V^{d}\left(y'\right)\right].$$

Therefore, we can rewrite the promise-keeping constraint as

$$\beta\mathbb{E}_{y'|y}\left(1-d\left(y'\right)\right)\left[\overline{V}^{nd}\left(b',y'\right)-V^{d}\left(y'\right)\right] \geq u\left(y\right)-u\left(y-b+b'q\right). \tag{A.8}$$

Step 4: Solve a relaxed version of program $P_{\underline{q}}$. Step 4.1: Determine the setup. We focus on a relaxed version of the problem. We will allow the default rule to be $d\left(y'\right) \in [0,1]$ for all y'. Given the state variables $\left(b,y,b'\right)$ the relaxed problem is a convex minimization program in the space $\left(q,d\left(\cdot\right)\right) \in \left[0,\frac{1}{1+r}\right] \times \mathbb{D}\left(Y\right)$, where:

$$\mathbb{D}\left(Y\right) \equiv \left\{d: Y \to [0,1] \text{ such that } d\left(y'\right)=1 \text{ for all } y' \notin R\left(b'\right)\right\}$$

is a convex set of default functions. In addition, we include a constraint for prices

$$q \geq \frac{\mathbb{E}_{y'|y}\left[1-d\left(y'\right)\right]}{1+r}.$$

The intuition for this last constraint is that $d\left(y'\right) = 1$ has to be feasible in the relaxed problem. The Lagrangian of the relaxed program is then:

$$\mathcal{L}\left(q, \delta\left(\cdot\right)\right) = q + \mu\left(-q + \frac{\mathbb{E}_{y'|y}\left[1 - d\left(y'\right)\right]}{1+r}\right) +$$
$$\lambda\left(u\left(y\right) - u\left(y - b + b'q\right) - \beta\mathbb{E}_{y'|y}\left[1 - d\left(y'\right)\right]\left(1 - d\left(y'\right)\right)\left[\overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right)\right]\right).$$

Step 4.2: FOC's point by point. The optimal default rule $d\left(\cdot\right)$ must minimize the Lagrangian $\mathcal{L}$, given the multipliers $\left(\mu, \lambda\right)$, where $\left(\mu, \lambda\right) \geq 0$. Note that for $y' \in R\left(b'\right)$, any $d \in \left[0, 1\right]$ is incentive constraint feasible, and

$$\frac{\partial \mathcal{L}}{\partial d\left(y'\right)} = \left(-\frac{\mu}{1+r} + \lambda\beta\left[\overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right)\right]\right) dF\left(y' \mid y\right)$$

where $dF\left(y' \mid y\right)$ is the "conditional probability" of state y' given y. Therefore, because it is a linear programming program, the solution is in the corners (if it is not in the corners, it has the same value in the interior); then, the values of y' such that the country does not default are given by

$$d\left(y'\right) = 0 \iff \lambda\left[\overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right)\right] > \frac{\mu}{\beta\left(1+r\right)}. \tag{A.9}$$

Step 4.3: Ensuring that $\lambda > 0$ in the optimum. Suppose that $\lambda = 0$; then $d\left(y'\right) = 1$ for all $y' \in Y$ satisfies the IC and the price constraint. Therefore, the minimum price is

$$q \geq \frac{1-1}{1+r} = 0.$$

Therefore, the minimizer will be zero, $q = 0$. However, this minimizer, $q = 0$, will not meet the promise-keeping constraint. Formally,

$$\beta \int V^{d}\left(y'\right) dF\left(y' \mid y\right) - \beta\mathbb{E}_{y'|y}V^{d}(y') - u\left(y\right) + u\left(y - b\right) =$$

$$= \beta\left(\mathbb{E}_{y'|y}V^{d}(y') - \mathbb{E}_{y'|y}V^{d}(y')\right) + u\left(y - b\right) - u\left(y\right) = u\left(y - b\right) - u\left(y\right) < 0.$$

This equation implies that $\lambda > 0$. Note that $\lambda > 0$ implies that $\underline{q}\left(b, y, b'\right) > 0$.

Step 4.4: Find γ such that $d\left(y'\right) = 0 \iff \overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right) + \gamma$. To do so, we define:

$$\gamma := \frac{\mu}{\lambda\beta\left(1+r\right)}.$$

Then, (A.9) implies that:

$$d\left(y'\right) = 0 \iff \overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right) + \gamma,$$

which is what we wanted to show.

How we compute γ? Step 5.1. Develop an equation for γ . Aided by this characterization, according to the promise -keeping constraint, we have an equation for γ that is a function of the states

$$\beta \int_{\overline{V}^{nd}(b',y') \geq V^{d}(y')+\gamma} \left[\overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right)\right] dF\left(y' \mid y\right) = u\left(y\right) - u\left(y - b + b'q\right) \quad \text{(A.10)}$$

where

$$q = \frac{\Pr\left(\overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right) + \gamma\right)}{1+r}. \quad \text{(A.11)}$$

Define

$$\Delta^{nd}(b', y') := \overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right).$$

So,

$$q = \frac{\hat{F}\left(\Delta^{nd}(b', y') \geq \gamma\right)}{1+r}$$

where $\hat{F}$ is the probability distribution of the random variable $\Delta^{nd}(b', y')$.

Step 5.2. Ensuring that the solution γ is well defined. Define the function

$$G\left(\gamma\right) = \beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}\left(\Delta^{nd} \mid y\right) - u\left(y\right) + u\left(y - b + b'\frac{1 - \hat{F}\left(\gamma \mid y\right)}{1+r}\right).$$

First, note that G is weakly decreasing in γ such that $G\left(0\right) > 0$ (from the assumption $\overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right) > 0$) and $\lim_{\gamma \to \infty} G\left(\gamma\right) = u\left(y - b\right) - u\left(y\right) < 0$. Second, note that G is right continuous in γ. These two observations imply that we can find a minimum $\gamma : G\left(\gamma\right) \geq 0$. If income is an absolutely continuous random variable, then $G\left(\cdot\right)$ is strictly decreasing and continuous, implying the existence of a unique γ such that $G\left(\gamma\right) = 0$. This process provides the solution to the price minimization problem. □

B Sunspot Proofs

Proof. Proposition 3. *Step 1: Necessity.*($\Longrightarrow$). Suppose there is an equilibrium strategy σ such that $h_m^{t+1} \in \mathcal{H}\left(\sigma\right)$. This strategy that the government optimally decided not to default at period t, which implies the following:

$$\int_0^1 \left[u\left(y_t - b_t + q^\sigma\left(h_m^{t+1},\zeta_t\right)b_{t+1}\right) + \beta V^\sigma\left(h_m^{t+1},\zeta_t\right)\right]d\zeta_t \geq u\left(y_t\right) + \beta\mathbb{E}_{y_{t+1}|y_t}V^d(y_{t+1}) \tag{B.1}$$

Recall that $\mathcal{E}\left(y_t, b_{t+1}\right)$ is the set of equilibrium payoffs of the game.[31] Since σ is an SPE, because of self generation, it holds that for all sunspot realizations $\zeta_t \in [0,1]$:

$$\left(V^\sigma\left(h_m^{t+1},\zeta_t\right), q^\sigma\left(h_m^{t+1},\zeta_t\right)\right) \in \mathcal{E}\left(y_t, b_{t+1}\right).$$

This further implies two things:

a. $q^\sigma\left(h_m^{t+1},\zeta_t\right) \in \left[0, \overline{q}\left(y_t, b_{t+1}\right)\right]$ (i.e., it delivers equilibrium prices)

b. $V^\sigma\left(h_m^{t+1},\zeta_t\right) \leq \overline{v}\left(y_t, b_{t+1}, q^\sigma\left(h_m^{t+1},\zeta_t\right)\right)$. This occurs because $\overline{v}$ is the maximum possible continuation value given the price realization $q = q^\sigma\left(h_m^{t+1},\zeta_t\right)$.

The price distribution given by σ can be defined by measure Q over measurable sets $A \subseteq \mathbb{R}_+$, as in:

$$Q\left(A\right) = \int_0^1 \mathbf{1}\left\{q^\sigma\left(h_m^{t+1},\zeta_t\right) \in A\right\}d\zeta_t = \Pr\left\{\zeta : q^\sigma\left(h_m^{t+1},\zeta_t\right) \in A\right\}.$$

Note that condition (a) shows that $Supp\left(Q\right) \subseteq \left[0, \overline{q}\left(y_t, b_{t+1}\right)\right]$. By changing the integration variables in B.1, using the definitions above and because of conditions (a) and (b):

$$\begin{aligned}\int_0^{\overline{q}(y_t,b_{t+1})}\left[u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right]dQ\left(\hat{q}\right) &\geq \int_0^1 \left[u\left(y_t - b_t + q^\sigma\left(h_m^{t+1},\zeta_t\right)b_{t+1}\right) + \beta V^\sigma\left(h_m^{t+1},\zeta_t\right)\right]d\zeta_t \\ &\geq u\left(y_t\right) + \beta\mathbb{E}_{y_{t+1}|y_t}V^d(y_{t+1});\end{aligned}$$

which proves the desired result.

Step 2: Sufficiency ($\Longleftarrow$). Suppose that Q is an equilibrium consistent distribution with

[31] In the Online Appendix Section D we define the equilibrium value correspondence and show how it can be computed. To make this proof self contained, we repeat the definition here:

$$\mathcal{E}\left(y_-, b\right) =: \left\{\left(v, q_-\right) \in \mathbb{R}_2 : \exists\sigma \in \Sigma^*\left(y_-, b\right) : \left[\begin{array}{c} v = \mathbb{E}\left\{\sum_{t=0}^\infty u\left(c^\sigma\left(h^t\right)\right)\right\} \\ c_t = y_t - b_t + q_t^{\sigma_m}b_{t+1} \\ b_0 = b \\ q_- = \frac{\mathbb{E}_{y|y_-}(1-d_0(y))}{1+r}\end{array}\right]\right\}.$$

This set has the utility values for the government and prices for the investors that can be obtained in a subgame perfect equilibrium, given an initial seed value y_-, and initial bonds b. Note that in the model of sovereign debt, we know that the set of prices is $[0, \overline{q}(y_-, b)]$ and the set of values is $[V^{aut}(y_-), \overline{\mathbb{V}}(y_-, b)]$.

the conditional density function F_Q. Let:

$$\sigma^*\left(y_t, b_{t+1}, q\right) \in \underset{\sigma \in \Sigma^*\left(y_t, b_{t+1}\right)}{\operatorname{argmax}} V^\sigma\left(h^0\right) \text{ s.t. } q_0^\sigma \leq q \,.$$

Note that by definition, $\sigma^*\left(y_t, b_{t+1}, q\right)$ is a strategy that achieves the continuation value $\overline{v}\left(y_t, b_{t+1}, q\right)$. As we show in the Online Appendix, Section D, the constraint in this problem, $q_0^\sigma \leq q$, is binding. Because h_m^{t+1} is an equilibrium consistent history, we know there exists an equilibrium profile $\hat{\sigma}$ such that $h_m^{t+1} \in \mathcal{H}\left(\hat{\sigma}\right)$. For histories h' successors of histories $h^{t+1} = \left(h^t, d_t, \hat{b}_{t+1}, \zeta_t, \hat{q}_t\right)$ we define the profile σ as:

$$\sigma\left(h'\right) = \begin{cases} \sigma^d\left(h'\right) & \text{if } d_t = 1 \text{or } \hat{b}_{t+1} \neq b_{t+1} \text{ or } \hat{q}_t \notin \left[0, \overline{q}\left(y_t, b_{t+1}\right)\right] \\ \sigma^*\left(y_t, b_{t+1}, \hat{q}_t\right)\left(h' \sim h^{t+1}\right) & \text{otherwise,} \end{cases}$$

and for histories $h' = \left(h^t, d_t = 0, b_{t+1}, \zeta_t\right)$, let

$$q^\sigma\left(h^{t+1}, y_t, d_t, b_{t+1}, \zeta_t\right) = F_Q^{-1}\left(\zeta_t\right)$$

where $F_Q\left(q\right) = Q\left(\hat{q}\right)$ is the cumulative distribution function of distribution Q and $F_Q^{-1}\left(\zeta\right) = \inf\left\{x \in \mathbb{R} : F_Q\left(q\right) \geq \zeta\right\}$ is its inverse. It will be optimal not to default at t (if we follow strategy σ for all successor nodes) if:

$$\int_0^1 \left[u\left(y_t - b_t + F_Q^{-1}\left(\zeta\right) b_{t+1}\right) + \beta V^\sigma\left(b_{t+1}, \zeta\right)\right] d\zeta \geq u\left(y_t\right) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1}) \iff$$

$$\int_0^{\overline{q}\left(y_t, b_{t+1}\right)} \left[u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right] dQ\left(\hat{q}\right) \geq u\left(y_t\right) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1}), \tag{B.2}$$

using the classical result that $F_Q^{-1}\left(\zeta\right) =_d Q$ if $\zeta \sim \text{Uniform}\left[0,1\right]$, then $V^\sigma\left(h'\right) = V\left(\sigma^*\left(h'\right)\right) = \overline{v}\left(y_t, b_{t+1}, q_t\right)$ according to the definition of σ. Condition B.1 is satisfied, and $Supp\left(Q\right) \subseteq \left[0, \overline{q}\left(y_t, b_{t+1}\right)\right]$ implies that if the government follows profile σ, then h is also on the path of σ, and σ is indeed a Nash equilibrium at such histories (because both σ^d and $\sigma^*\left(y_t, b_{t+1}, \hat{q}\right)$ are subgame perfect profiles). Finally, for histories $h' \not\succ h^t$ we define $\sigma\left(h'\right) = \hat{\sigma}\left(h'\right)$. Therefore, $\sigma\left(h'\right)$ is an SPE profile (since it is a Nash equilibrium at every possible history) and generates $h = \left(h^t, d_t = 0, b_{t+1}\right)$ on its path. □

Proof. Proposition 4. *Step 1: Determine the upper bound for probability of $q = 0$.* To construct the maximum equilibrium consistent probability such that $q_t = 0$ after history h_m^{t+1}, we need to make the promise-keeping constraint as relaxed as possible. We do this by focusing on probability distributions $\underline{Q}$ that are binary. These distributions place positive probability only on the worst equilibrium price and best equilibrium prices. We denote the, largest, probability of a price equal to zero as $\underline{Q}(\hat{q}=0)$. As a consequence, $1-\underline{Q}(\hat{q}=0)$ is the, lowest, probability of the best equilibrium consistent price. The IC constraint for this distribution is now:

$$\underline{Q}(\hat{q}=0)\left[u(y_t-b_t)+\beta\mathbb{E}_{y_{t+1}|y_y}V^d(y_{t+1})\right]+\left(1-\underline{Q}(\hat{q}=0)\right)\left[\overline{V}^{nd}(b_t,y_t,b_{t+1})\right]=V^d(y_t).$$

Then

$$\underline{Q}(\hat{q}=0)=\frac{\Delta^{nd}(b_t,y_t,b_{t+1})}{\Delta^{nd}(b_t,y_t,b_{t+1})+u(y_t)-u(y_t-b_t)}<1,$$

where $\Delta^{nd}(\cdot)$ denotes the maximum utility difference between not defaulting and defaulting (under the best equilibrium)

$$\Delta^{nd}(b_t,y_t,b_{t+1})\equiv\overline{V}^{nd}(b_t,y_t,b_{t+1})-V^d(y_t).$$

Thus, the probability of $q=0$ is bounded away from 1 from an ex-ante perspective (i.e. before the sunspot is realized, but after the government decision's decision has been made). Therefore, we obtain a history dependent bound on the probability of a financial crisis.

Step 2: Determine the upper bound for $q=\hat{q}$. First, we determine the upper bound for general $\hat{q}<\underline{q}(b_t,y_t,b_{t+1})$. Here, we use the same strategy: let $p=\Pr(\zeta_t:q(\zeta_t)\leq\hat{q})$. Using the same strategy as before, to obtain a less binding IC constraint for the government, we need to maximize equilibrium utility for $q(\zeta_t)>\hat{q}$. Thus, we consider equilibria that assigns the best continuation equilibria in this case (to make the IC of the government as flexible as possible). Consider the continuation equilibria where $q(\zeta_t)=\overline{q}(y_t,b_{t+1})$ and $v(\zeta_t)=\overline{\mathbb{V}}(y_t,b_{t+1})$ (the fact that this corresponds to an actual equilibria is easy to check). With a reasoning that is similar to the one in Step 1, we see that focusing on equilibria that have support $q(\zeta_t)\in\{\hat{q},\overline{q}(y_t,b_{t+1})\}$ makes the government's IC as flexible as possible. This, because the utility of the government is increasing in $\hat{q}$ and moreover, $\overline{v}(y_-,b,\hat{q})$ (the greatest continuation utility consistent with $q\leq\hat{q}$) is also increasing in $\hat{q}$ as we saw before. Therefore, if p is the maximum such probability, then we must have:

$$p\left[u(y_t-b_t+\hat{q}b_{t+1})+\beta\overline{v}(y_t,b_{t+1},\hat{q})\right]+(1-p)V^{nd}(b_t,y_t,b_{t+1})\geq V^d(y_t)\iff$$

$$p \leq \frac{\Delta^{nd}\left(b_t, y_t, b_{t+1}\right)}{V^d\left(y_t\right)-\left[u\left(y_t-b_t+\hat{q} b_{t+1}\right)+\beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right]+\Delta^{nd}\left(b_t, y_t, b_{t+1}\right)}.$$

Note that this is not an innocuous constraint only when the right hand side is less than 1, which happens only when

$$u\left(y_t-b_t+\hat{q} b_{t+1}\right)+\beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right) \geq V^d\left(y_t\right).$$

As we argued before,

$$\hat{q} \geq \underline{q}\left(b_t, y_t, b_{t+1}\right)$$

where the last inequality comes from the characterization of $\underline{q}\left(b_t, y_t, b_{t+1}\right)$. □

Proof. Proposition 5. We already know that $\max E\left(b_t, y_t, b_{t+1}\right)=\overline{q}\left(y_t, b_{t+1}\right)$ since the Dirac distribution $\overline{P}$ over $q=\overline{q}\left(y_t, b_{t+1}\right)$ is equilibrium consistent. In the same way, we also know that the Dirac distribution $\hat{Q}$ that assigns probability 1 to $q=\underline{q}\left(b_t, y_t, b_{t+1}\right)$ is equilibrium consistent; this distribution corresponds to a case where both investors and the government ignore the realization of the correlated device, and the characterization of $\underline{q}\left(\cdot\right)$ is exactly the only price that satisfies

$$u\left(y_t-b_t+\underline{q}\left(b_t, y_t, b_{t+1}\right)\ b_{t+1}\right)+\beta \overline{v}\left(y_t, b_{t+1}, \underline{q}\left(b_t, y_t, b_{t+1}\right)\right)=V^d\left(y_t\right).$$

In the Online Appendix, Section D, Lemma 3, we show that $\overline{v}\left(y_{-}, b, q\right)$ is a concave function in q, which together with the fact that u is strictly concave and $b'>0$ implies that the function

$$H(q):=u\left(y_t-b_t+q b_{t+1}\right)+\beta \overline{v}\left(y_t, b_{t+1}, q\right)$$

is strictly concave in q. For any distribution $Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$, let $\mathbb{E}_Q(q)=\int \hat{q} dQ(\hat{q})$. Jensen's inequality then implies that

$$\begin{aligned} u\left(y_t-b_t+\mathbb{E}_Q(q)\, b_{t+1}\right)+\beta \overline{v}\left(y_t, b_{t+1}, \mathbb{E}_Q(q)\right) & \underbrace{\geq}_{(1)} \int\left[u\left(y_t-b_t+\hat{q} b_{t+1}\right)+\beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right] dQ(\hat{q}) \\ & \underbrace{\geq}_{(2)} V^d\left(y_t\right) \end{aligned}$$

with strict inequality in (1) if Q is not a Dirac distribution. Then, the definition of $\underline{q}(b_t, y_t, b_{t+1})$ implies that for any distribution $Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ we have that:

$$\mathbb{E}_Q(q) \geq \underline{q}\left(b_t, y_t, b_{t+1}\right);$$

therefore, the minimum expected value is exactly $\underline{q}(b_t, y_t, b_{t+1})$, which is achieved uniquely at the Dirac distribution $\hat{Q}$ (because of the strict concavity of $u(\cdot)$). Finally, knowing that E is naturally a convex set, we then obtain

$$\begin{aligned} E(b_t, y_t, b_{t+1}) &= \left[\min_{Q \in \mathcal{Q}(b_t, y_t, b_{t+1})} \int \hat{q} dQ(\hat{q}), \max_{Q \in \mathcal{Q}(b_t, y_t, b_{t+1})} \int \hat{q} dQ(\hat{q})\right] \\ &= \left[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(b_t, y_t, b_{t+1})\right] \end{aligned}$$

which is what we wanted to show. □

Proof. Proposition 6. *Step 1: Determine the bounds for General Random Variables.* To show the bounds on the variance, we rely on the fact that for any random variable X with support in $[a, b] \subseteq \mathbb{R}$ and mean $\mathbb{E}(X) = \mu$, it holds that:

$$\mathbb{V}ar(X) \leq \mu(b + a - \mu) - ab.$$

Moreover, this upper bound in the variance is achieved by a binary distribution P_μ over $\{a, b\}$, with $P_\mu(a) = (b - \mu) / (b - a)$, and of course, $P_\mu(b) = 1 - P_\mu(a)$.

Step 2: Are these bounds Equilibrium Consistent? It Depends. Since the price realization must have support on $[0, \overline{q}(y_t, b_{t+1})]$, after history h_m^t, according to Proposition 3, we know that if $Q : \mathbb{E}_Q(q_t) = \mu$ then $\mathbb{V}_Q(q_t) \leq \mu(\overline{q}(y_t, b_{t+1}) - \mu)$; this bound is achieved by distribution Q_μ with $Q_\mu(0) = \frac{\overline{q} - \mu}{\overline{q}}$. However, this particular distribution may not be equilibrium consistent since it may violate the ex-ante IC for no default, condition (4.1),

$$\int_0^{\overline{q}(y_t, b_{t+1})} [u(y_t - b_t + q b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, q)] \, dQ_\mu(q) \geq V^d(y_t).$$

Whether this constraint is violated or not will depend on the particular value of $\mu \in \left[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})\right]$. We define $q^* = \underline{Q}(0) \times 0 + \left(1 - \underline{Q}(0)\right) \overline{q}$.

Step 3: Case 1. IC is not binding for the candidate distribution if the mean is high enough. We first show that if $\mathbb{E}_Q(q_t) = \mu \geq q^*$, then any distribution $Q \in \Delta([0, \overline{q}])$ with $\mathbb{E}_P(q_t) = \mu$ also satisfies 4.1, and hence the maximum variance is achieved precisely at $\mu(\overline{q} - \mu)$. We now show this. We define

$$D\left(h_m^{t+1}, q_t\right) := u(y_t - b_t + q_t b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, q_t) - V^d(y_t);$$

as the difference between the best continuation given a price q_t and history h_m^{t+1}, and the worst equilibrium. Remember that $q^* = \underline{Q}(0) \times 0 + \left(1 - \underline{Q}(0)\right) \overline{q}$. Using the definition of $\underline{Q}(0)$, it can be shown that

$$\underline{Q}(0) = \frac{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)}{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t) + u(y_t) - u(y_t - b_t)}.$$

Thus, using the definition of $D\left(h_m^{t+1}, q_t\right)$ at q^*:

$$\begin{aligned}
D\left(h_m^{t+1}, q^*\right) &= D\left(h_m^{t+1}, \underline{Q}(0) \times 0 + \left(1 - \underline{Q}(0)\right)\overline{q}\right) \\
&> \underline{Q}(0)\, D\left(h_m^{t+1}, 0\right) + \left(1 - \underline{Q}(0)\right) D\left(h_m^t, \overline{q}\right) \\
&= \frac{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)}{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t) + u(y_t) - u(y_t - b_t)}\left[u(y_t - b_t) - u(y_t)\right] + \\
&\frac{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)}{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t) + u(y_t) - u(y_t - b_t)}\left[u(y_t) - u(y_t - b_t)\right] \\
&= 0.
\end{aligned}$$

Therefore, by the concavity of $\overline{v}(y_t, b_{t+1}, q_t)$,

$$\begin{aligned}
\int D\left(h_m^{t+1}, q_t\right) dQ(q_t) \quad &\underbrace{\geq}_{D \text{ is concave in } q} \quad D\left(h_m^{t+1}, \mu_Q\right) \\
&\underbrace{\geq}_{D \text{ is increasing in } q} \quad D\left(h_m^{t+1}, q^*\right) \\
&> 0.
\end{aligned}$$

Thus, when $\mu \geq q^*$ then $\overline{\mathbb{V}ar}(q_t) = \mu\left(\overline{q}(y_t, b_{t+1}) - \mu\right)$.

We also check that $q^* > \underline{q}$. This holds because $D\left(h_m^{t+1}, \underline{q}\right) = 0$ and $D\left(h_m^{t+1}, q^*\right) > \underline{Q}(0)\, D\left(h_m^{t+1}, 0\right) + \left(1 - \underline{Q}(0)\right) D\left(h_m^{t+1}, \overline{q}\right) = 0$, which then implies that $q^* > \underline{q}$ (because D is strictly increasing in q).

Step 4.1: Case 2. Proposal Violates IC for a Low Mean. We also show that if $Q : \mathbb{E}_Q(q_t) = \mu < q^*$, then the distribution Q_μ defined as $Q_\mu(0) = \frac{\overline{q} - \mu}{\overline{q}}$ and $1 - Q_\mu(0)$ violates the ex ante no default incentive constraint 4.1. This follows because:

$$
\begin{aligned}
\mathbb{E}_{Q_\mu}\left[D\left(h_m^{t+1},q_t\right)\right] &= \left(1-\frac{\mu}{\overline{q}}\right)D\left(h_m^{t+1},0\right)+\frac{\mu}{\overline{q}}D\left(h_m^{t+1},\overline{q}\right) \\
&= D\left(h_m^{t+1},0\right)+\frac{\mu}{\overline{q}}\left[D\left(h_m^{t+1},\overline{q}\right)-D\left(h_m^{t+1},0\right)\right] \\
&= D\left(h_m^{t+1},0\right)+\frac{\mu}{\overline{q}}\left[D\left(h_m^{t+1},\overline{q}\right)-D\left(h_m^{t+1},0\right)\right] \\
&< D\left(h_m^{t+1},0\right)+\frac{\left(1-Q_\mu\left(0\right)\right)\overline{q}}{\overline{q}}\left[D\left(h_m^{t+1},\overline{q}\right)-D\left(h_m^{t+1},0\right)\right] \\
&= D\left(h_m^{t+1},0\right)+\frac{-D\left(h_m^{t+1},0\right)}{D\left(h_m^{t+1},\overline{q}\right)-D\left(h_m^{t+1},0\right)}\left[D\left(h_m^{t+1},\overline{q}\right)-D\left(h_m^{t+1},0\right)\right] \\
&= 0
\end{aligned}
$$

where we use that $\mu < q^*$ and the definition of $q^* = \left(1-\underline{Q}\left(0\right)\right)\overline{q}$. Thus:

$$
\mathbb{E}_{Q_\mu}\left[D\left(h_m^{t+1},q_t\right)\right] < 0.
$$

This implies that the candidate Q_μ is not an equilibrium consistent price distribution when $\mu < q^*$.

Step 4.2: A New Proposal. To show the second result, following Step 1, we know that we need to restrict attention to binary support distributions; because $D\left(h_m^{t+1},q_t\right)$ is concave, it is easy to show that the support that maximizes the variance (for a given expectation $\mu < q^*$) is $\left\{q_\mu,\overline{q}\right\}$ for some q_μ. Since the no default incentive constraint is binding and we also have a given expectation μ, we need to find q_μ and $\Pr\left(q_\mu\right)$ to solve the following system of equations:

$$
\begin{cases}
\Pr\left(q_\mu\right)q_\mu+\left(1-\Pr\left(q_\mu\right)\right)\overline{q}=\mu \\
\Pr\left(q_\mu\right)D\left(h,q_\mu\right)+\left(1-\Pr\left(q_\mu\right)\right)D\left(h,\overline{q}\right)=0.
\end{cases}
$$

Next, note that the second constraint (the no-default incentive constraint), given q_μ is the definition of the infimum distribution

$$
\underline{Q}\left(q_\mu\right)=D\left(h_m^{t+1},\overline{q}\right)/\left(D\left(h_m^{t+1},\overline{q}\right)-D\left(h_m^{t+1},q_\mu\right)\right)
$$

given in Proposition 4. Using this on the first equation, we obtain one equation in the unknown q_μ :

$$\underline{Q}(q_\mu) q_\mu + (1 - \underline{Q}(q_\mu)) \overline{q} = \mu \iff \frac{D(h_m^t, \overline{q}) - D(h_m^t, q_\mu)}{\overline{q} - q_\mu} = \frac{D(h_m^t, \overline{q})}{\overline{q} - \mu}. \tag{B.3}$$

Because $D(h_m^{t+1}, q)$ is increasing in q, the solution q_μ of equation B.3 is increasing in μ in the region where $\mu < q^*$. □

Proof. Corollary 2. This is true because the function

$$U(Q; b_t, y_t, b_{t+1}) = \int \{u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})\} \, dQ(q)$$

is strictly increasing in y_t and strictly decreasing in b_t, and the set can be rewritten as

$$\mathcal{Q}(b_t, y_t, b_{t+1}) = \left\{ Q \in \Delta([0, \overline{q}]) : U(Q; b_t, y_t, b_{t+1}) \geq V^d(y_t) \right\}.$$

The function $H(q) := u(y_t - b_t + q b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, q)$ is strictly increasing in q. Therefore, if Q'FOSDQ and $Q \in \mathcal{Q}(b_t, y_t, b_{t+1})$ then $\int H(q) \, dQ' \geq \int H(q) \, dQ \geq V^d(y_t)$. Finally we show that $\underline{Q}$ is not en equilibrium consistent distribution. By definition, equation 4.2 cannot be an equilibrium consistent price; this implies that the Lebesgue-stjeljes measure associated with $\underline{Q}(\cdot)$ has the property that $Supp(Q) = \left[0, \underline{q}(b_t, y_t, b_{t+1})\right]$ and $\underline{Q}(q = 0) = p_0 > 0$, which implies that

$$\begin{aligned} \int_0^{\overline{q}(y_t, b_{t+1})} \{u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})\} \, d\underline{Q}(\hat{q}) &< u\left(y_t - b_t + \underline{q}(\cdot) b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \underline{q}(\cdot)\right) \\ &= V^d(y_t) \end{aligned}$$

where the last equation comes from the definition of $\underline{q}(\cdot)$ and the function $H(\hat{q}) \equiv u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})$ is strictly increasing in $\hat{q}$. □

Online Appendix to "Robust Predictions in Dynamic Policy Games"

Juan Passadore and Juan Xandri

C Multipe Equilibrium in Eaton and Gersovitz (1981)

This appendix studies equilibrium multiplicity in the model proposed in 2. In particular, we characterize the best and worst equilibrium prices (Propositions 9 and 11); the whole set of equilibria (Proposition 10); we provide sufficient conditions for equilibrium multiplicity (Proposition 12); we provide a numerical example for multiplicity; and finally, we discuss which are the implications for deviations from our stylized setting for equilibrium multiplicity. Our results complement the results in Auclert and Rognlie (2016); their paper shows uniqueness in the Eaton and Gersovitz (1981) when the government can save and savings are valued and extends the result for costs of default and the possibility of re-entry.

Preliminaries. For any history h_m^{t+1} we consider the highest and lowest prices

$$\overline{q}(h_m^{t+1}) := \max_{\sigma \in \Sigma^*(h_m^{t+1})} q_m \left(h_m^{t+1} \right)$$

$$\underline{q}(h_m^{t+1}) := \min_{\sigma \in \Sigma^*(h_m^{t+1})} q_m \left(h_m^{t+1} \right).$$

where $\Sigma^*(h_m^{t+1})$ is the set of equilibria after history h_m^{t+1}. As it will be clear from this section, the set $\Sigma^*(h_m^{t+1})$ is equal to $\Sigma^*(y_t, b_{t+1})$; i.e, the set is pinned down only by y_t, b_{t+1}. The best and worst equilibria turn out to be Markov equilibria and we find conditions for multiplicity. The worst SPE price is zero, and the best SPE price is the one for the Markov equilibrium that is characterized on sovereign debt, such as Arellano (2008) and Aguiar and Gopinath (2006). Our analysis may be of independent interest, because it describes the conditions under which there are multiple Markov equilibria in a sovereign debt model, similar tot he one proposed in Eaton and Gersovitz (1981). The importance of this result is that it opens up the possibility of confidence crises in models as in Eaton and Gersovitz (1981). Thus, confidence crises are not necessarily a special feature of the timing in Calvo (1988) and Cole and Kehoe (2000) but rather robust features in most models of sovereign debt. The lowest price $\underline{q}(h_m^{t+1})$ can be attained by using a fixed strategy for all histories h_m^{t+1}. This strategy will deliver the utility level of autarky for the government. Thus, the lowest price is associated with the worst equilibrium, in terms of welfare. Likewise, the highest price $\overline{q}(h_m^{t+1})$ is associated with a different fixed strategy for all histories (the maximum is attained by the same σ for all h_m^{t+1}) and delivers the highest equilibrium level of utility for the government. Thus, the highest price is associated with the best equilibrium in terms of welfare.

C.1 Lowest Equilibrium Price and Worst Equilibrium

We start by showing that, after any history h_m^{t+1}, the lowest SPE is equal to zero. Denote by $\mathbf{B}$ the set of assets for the government. We assume that the government cannot save; i.e. $\mathbf{B} \geq 0$.

Proposition 9. $\mathbf{B}$ *denotes the set of assets for the goverment. Under the assumption of* $\mathbf{B} \geq 0$*, the lowest SPE price is equal to zero*

$$\underline{q}(h_m^{t+1}) = \underline{q}(y_t, b_{t+1}) = 0$$

and is associated with a Markov equilibrium that achieves the worst level of welfare.

When the government is confronted with a price of zero for its bonds in the present period and expects to face the same price in all future periods, it is best to default. The government cannot benefit from repaying the debt. The proof is simple. We need to show that defaulting after every history is an SPE. Because the game is continuous at infinity, we need to show that there are no profitable one shot deviations when the government uses this strategy. Note, first, that if the government uses a strategy of always defaulting, it is effectively in autarky. In history h_m^{t+1} with income y_t and debt b_t, the payoff of such a strategy is

$$u(y_t) + \frac{\beta}{1-\beta}\mathbb{E}_{y_{t+1}|y_t} u(y_{t+1}).$$

Note also that, a one shot deviation involving repayment today has associated utility of

$$u(y_t - b_t) + \frac{\beta}{1-\beta}\mathbb{E}_{y_{t+1}|y_t} u(y_{t+1}).$$

Thus, as long as b_{t+1} is non-negative, a one shot deviation of repayment is not profitable. Therefore, autarky is an SPE with an associated price of debt equal to zero.

C.2 Highest Equilibrium Price and Best Equilibrium

We now characterize the best SPE and show that it is the Markov equilibrium studied by the literature on sovereign debt. To find the worst equilibrium price, it was sufficient to use the definition of equilibrium and the one shot deviation principle. To find the best equilibrium price it will be necessary to find a characterization of equilibrium prices. Denote by $\overline{\mathbb{V}}(y_t, b_{t+1})$ the highest expected equilibrium payoff if the government enter period $t+1$ with bonds b_{t+1} and income in t was y_t. The next lemma provides a characterization of equilibrium outcomes.

Proposition 10. *$x_{t,m} = (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$ is an SPE outcome for history h_m^t if and only if the following conditions hold:*

a. The price is consistent

$$q_{t-1} = \frac{\mathbb{E}_{y_t|y_{t-1}}(1 - d_t(y_t))}{1+r}, \tag{C.1}$$

b. IC of the government

$$(1 - d(y_t))\left[u(y_t - b_t + \overline{q}(y_t, b_{t+1})b_{t+1}) + \beta\overline{\mathbb{V}}(y_t, b_{t+1})\right] + d(y_t)V^d(y_t) \geq V^d(y_t). \tag{C.2}$$

The proof is omitted; it is a particular case of the main result for the model without sunspots. Condition (C.1) states that the price q_{t-1} needs to be consistent with the default policy $d_t(\cdot)$. Condition (C.2) states that a policy $d_t(\cdot), b_{t+1}(\cdot)$ is implementable in an SPE if it is incentive compatible given that following the policy is rewarded with the best equilibrium and a deviation is punished with the worst equilibrium. The argument in the proof follows Abreu (1988). These two conditions are necessary and sufficient for an outcome to be part of an SPE.[32]

Markov Equilibrium. We now characterize the Markov equilibrium that is usually studied in the literature on sovereign debt. The value of a government that has the option to default is given by

$$\overline{\mathbb{V}}(y_-, b) = \mathbb{E}_{y|y_-}\left[\max\left\{\overline{V}^{nd}(b, y), V^d(y)\right\}\right]. \tag{C.3}$$

[32]Note that for any history (even those that are *inconsistent* with equilibria) SPE policies are a function of only one state: the debt that the government has to pay at time t (b_t). There are two reasons for this. First, the stock of debt summarizes the physical environment. Second, the value of the worst equilibrium depends only on the realized income.

This is the expected value of the maximum between not defaulting $\overline{V}^{nd}(b, y)$ and the value of defaulting $V^D(y)$. The value of not defaulting is given by

$$\overline{V}^{nd}(b, y) = \max_{b' \geq 0} u(y - b + q(y, b')b') + \beta \overline{\mathbb{V}}(y, b'). \tag{C.4}$$

That is, the government repays the debt and obtains a capital inflow (outflow), and from the budget constraint consumption is given by $y - b + q(y, b')b'$; in the next period, the government has the option to default on b' bonds. The value of defaulting is

$$V^d(y) = u(y) + \beta \frac{\mathbb{E}_{y'|y} u(y')}{1 - \beta}, \tag{C.5}$$

and is just the value of consuming income forever. These value functions define a default set

$$D(b) = \left\{ y \in Y : \overline{V}^{nd}(b, y) < V^d(y) \right\}. \tag{C.6}$$

A Markov Equilibrium (with states b, y) is a set of policy functions

$$(c(y, b), d(y, b), b'(y, b)),$$

a bond price function $q(y, b')$ and a default set $D(b)$ such that $c(y, b)$ satisfies the resource constraint; taking as given $q(y, b')$ the government bond policy maximizes $\overline{V}^{nd}$, and the bond price $q(y, b')$ is consistent with the default set

$$q(y, b') = \frac{1 - \int_{D(b')} dF(y' \mid y)}{1 + r}. \tag{C.7}$$

The next proposition states that the best Markov equilibrium is the best SPE

Proposition 11. *The best SPE is the best Markov equilibrium.*

Proof. According to lemma 10, the value of the best equilibrium is the expectation with respect to y_t, given y_{t-1}, and is given by

$$\max_{d_t, b_{t+1}} (1 - d_t) \left[u(y_t - b_t + \overline{q}(y_t, b_{t+1})b_{t+1}) + \beta \overline{\mathbb{V}}(y_t, b_{t+1}) \right] + d_t V^d(y_t).$$

Note that this is equal to the left hand side of (C.3). The key assumption for ensuring that the best SPE is the best Markov equilibrium is that the government is punished with permanent autarky after a default. $\square$

C.3 Multiplicity

Given that the worst equilibrium is autarky, a sufficient condition for the multiplicity of Markov equilibria is any condition that guarantees that the best Markov equilibria has positive debt capacity, which is a standard situation in quantitative sovereign debt models. In general some debt can be sustained as long as there is enough of a desire to smooth consumption, which will motivate the government to avoid default, at least for small debt levels. The following proposition provides a simple sufficient condition for this to be the case. We define $\mathcal{V}^{nd}(b, y; B, \frac{1}{1+r})$ as the value function when the government faces the risk free interest rate $q = \frac{1}{1+r}$; there is a borrowing limit B as in a standard Bewley incomplete market model. The government has the option to default. This value is not an upper bound on the possible values of the borrower because default introduces state contingency and might be valuable. Our next proposition, however, establishes conditions under which default does not take place.

Proposition 12. *Suppose that for all $b \in [0, B]$ and all $y \in \mathbf{Y}$*

$$\mathcal{V}^{nd}(b, y; B, \frac{1}{1+r}) \geq u(y) + \beta \mathbb{E}_{y'|y} V^d(y'). \tag{C.8}$$

Then multiple Markov equilibria exist.

Proof. If the government is confronted with $q = \frac{1}{1+r}$ for $b \leq B$, then condition (C.8) ensures that it will not want to default after any history, which justifies the risk free rate for $b \leq B$. An SPE can implicitly enforce the borrowing limit $b \leq B$ by triggering autarky and setting $q_t = 0$ if $b_{t+1} > B$ ever occurs. Since the debt issuance policy is optimal given the risk free rate, we have constructed an equilibrium. This proves that there is at least one SPE sustaining strictly positive debt and prices. The best equilibrium dominates this one and is Markov, as shown earlier, so it follows that there exists at least one strictly positive Markov equilibrium. Finally, note that we need to check condition (C.8) only for small values of B. However, the existence result then extends an SPE across the entire $\mathbf{B} = [0, \infty)$.[33] □

Example. Suppose there are two income shocks y_L and y_H that follow a Markov chain (a special case is the i.i.d. case). For this case, λ_i denotes the probability of transitioning from state i to state $j \neq i$. We construct an equilibrium where debt is risk free, and the government goes into debt B, stays there as long as its income is low, repays the debt and

[33] Indeed, it is useful to consider small B and take the limit, which then requires checking only a local condition. The following example illustrates this condition.

remains debt free when income is high. Conditional on not defaulting, this bang bang solution is optimal for small enough B. To investigate whether default is avoided, we must compute the values

$$
\begin{aligned}
v_{BL} &= u(y_L + (R-1)B) + \beta\left(\lambda_L v_{BH} + (1-\lambda_L)v_{BL}\right) \\
v_{BH} &= u(y_H - RB) + \beta\left(\lambda_H v_{0L} + (1-\lambda_H)v_{0H}\right) \\
v_{0L} &= u(y_L + B) + \beta\left(\lambda_L v_{BH} + (1-\lambda_L)v_{BL}\right) \\
v_{0H} &= u(y_H) + \beta\left(\lambda_H v_{0L} + (1-\lambda_H)v_{0H}\right)
\end{aligned}
$$

where $R = 1 + r$. We write the solution to this system as a function of B. To guarantee that the government does not default in any state, we need to check that $v_{BL}(B) \geq v^{aut}$, $v_{BH}(B) \geq v^{aut}$, $v_{0L}(B) \geq v_L^{aut}$ and $v_{0H}(B) \geq v_H^{aut}$ (some of these conditions can be shown to be redundant). The following proposition provide a simple parametric assumption in which the sufficient conditions hold.

Proposition 13. *A sufficient condition for $v_{BL} \geq v^{aut}, v_{BH} \geq v^{aut}, v_{0L} \geq v_L^{aut}, v_{0H} \geq v_H^{aut}$ that holds for some $B > 0$ is $v'_{BL}(0) > 0, v'_{BH}(0) > 0$. When $\lambda_H = \lambda_L = 1$ this condition simplifies to $\beta u'(y_L) > Ru'(y_H)$.*

Note that the simple condition with $\lambda_H = \lambda_L = 1$ is met when u is sufficiently concave or if β is sufficiently close to 1. These conditions ensure that the value from consumption smoothing is high enough to sustain debt.

Proof. Note that we can rewrite the system of Bellman equations as

$$A.v(B) = u(B)$$

Thus, a condition in primitives is

$$v'(0) = A^{-1}u'(0) \geq 0$$

For the special case where $\lambda = 1$, note that

$$
\begin{aligned}
v_{BH} &= \frac{1}{1-\beta^2}\left(u(y_H - RB) + \beta u(y_L + B)\right) \\
v_{0L} &= u(y_L + B) + \beta v_{BH}
\end{aligned}
$$

Then, $v'_{BH}(0) > 0$ implies that $v'_{0L}(0) > 0$. A sufficient condition is $\beta u'(y_L) > Ru'(y_H)$. The intuition is that, the government is credit constrained in the low state, with no debt,

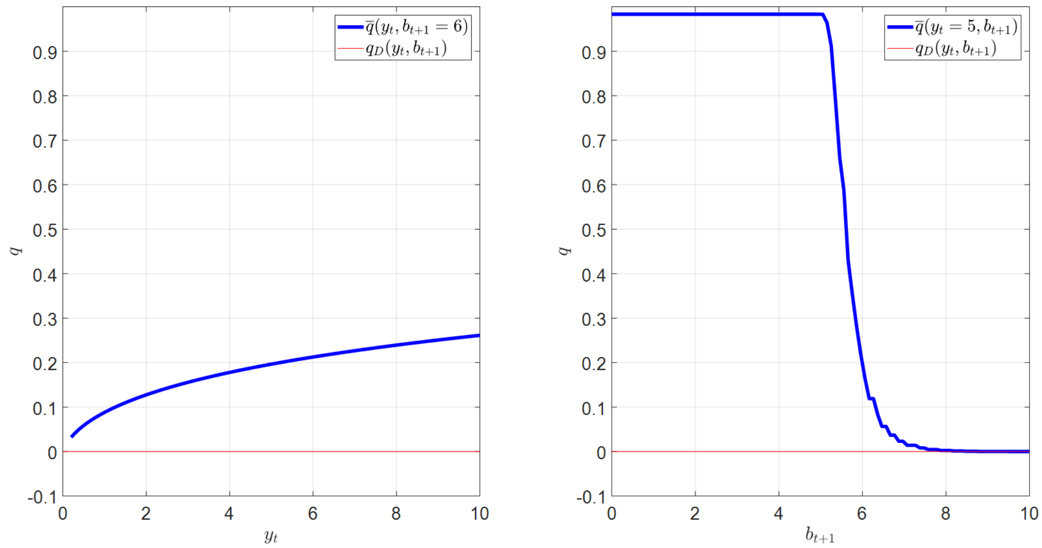

Figure 6: This figure plots the best and worst equilibrium pricing functions: $\overline{q}(y, b')$ and the worst equilibrium price equal to zero.

and is willing to tradeoff and have lower consumption in the high state. $\square$

A Numerical Illustration. We now numerically illustrate equilibrium multiplicity. The process for log output is given by $\log y_t = \mu + \rho_y \log y_{t-1} + \sigma_y \epsilon_t$ where $\mu = 0.75$. $\sigma_y = 0.3025$, $\rho_y = 0.0945$. The parameters are the same that we use in the main calibration: the discount factor $\beta = 0.953$, CRRA utility with relative risk aversion $\gamma_{RRA} = 2$ and the risk-free interest rate $r = 0.017$. Figure C.3 presents the results. The value functions are the ones in equations (C.3) to (C.7) and the price is given by (C.7). The worst equilibrium has the value of autarky and a price of zero. The best equilibrium is the one studied in quantitative models with short-term debt as in Arellano (2008). Our case is different to Arellano (2008) because there is permanent exclusion after default and there are no direct costs of default. We plot the two price functions, the one of the best equilibrium and the other one is equal to zero (autarky). As it is clear from the right panel, in the best equilibrium for low levels of debt and income debt is risk-free. As we increase the level of debt, the price drops. The price drop is sharp, as is most models with short-term debt.

C.4 Discussion

We close this section with a discussion of conditions under which there is unique or multiple equilibria. First, notice, that sunspots are not needed to generate multiple equilibria.

Sunspots may act as a coordinating device to select a particular equilibrium, but we did not use any property of output as a coordinating device to show that either autarky or the best equilibrium are equilibria or that they are different; i.e. we did not use them in any part of Propositions 9, 10, and 12.

Second, as we mentioned before, things are different when the government is allowed to save before default and the punishment is autarky, including exclusion from saving. Under this combination of assumptions, the government might want to repay small amounts of debt to maintain the option to save in the future. As a result, autarky is no longer an equilibrium. Furthermore, a unique subgame perfect equilibrium prevails, as shown by Auclert and Rognlie (2014). Note however, that the government needs to value savings. If savings are not valued, which is a parametric assumption, that means that the value of smoothing consumption with savings is the same as the value of autarky, a condition that is micro-founded in Amador (2013), then autarky will again we an equilibrium; it is easy to modify the proof of Proposition 9 for this case. Furthermore, one can find examples in which savings are not valued and the sufficient conditions for multiplicity of Proposition 12 hold. Finally, note that non-uniqueness holds given a non-equilibrium punishment.

Third, whether or not there are direct costs of default matter for equilibrium multiplicity. For autarky to be an equilibrium, it has to be a dominant strategy to default on any amount of debt that it is allowed to hold if they face a zero price, i.e. to default for all $b \in \mathbf{B}$ if $q = 0$. With default costs, the value of defaulting is lower. Therefore, as with the case of savings, if these costs are large, the government might want to repay small amounts of debt even though the market is offering a zero price of debt in all future period, because the cost of default is too high. Thus, we need to increase the static gain of defaulting for any history. A sufficient condition would then be that $\mathbf{B} > 0$. The lower bound on debt will be increasing in the magnitude of the output costs of default. In a related paper Stangebye (2018) studies a case that is related though different and numerically finds multiple equilibria. There are output costs of default but, different from the case in this paper there is long term debt which provides additional forces for equilibrium multiplicity.

D Characterization of $\overline{v}\left(y_-, b, q_-\right)$

In this section we characterize the best ex-post continuation value when the income realized is y_- and b bonds are issued at price q_-; i.e.

$$\overline{v}\left(y_-, b, q_-\right) := \max_{\sigma \in \Sigma^*(y_-,b)} V\left(\sigma \mid y_-, b_0 = b, q_-\right).$$

The procedure consists of two steps. In the first step, we characterize the set of equilibrium payoffs $\mathcal{E}\left(y_-, b\right)$:the values for the government and the prices for the investors. We base our characterization on the concept of self-generation, introduced in Abreu et al. (1990) which has applications for monetary policy Chang (1998b), capital taxation Phelan and Stacchetti (2001) and sovereign lending Atkeson (1991). In the second step, using the set of equilibrium values and prices, $\mathcal{E}\left(y_-, b\right)$, we show how to compute $\overline{v}\left(y_-, b, q\right)$.

D.1 Step 1: Characterizing the Equilibrium Set $\mathcal{E}\left(y_-, b\right)$

We define the equilibrium value correspondence as

$$\mathcal{E}\left(y_-, b\right) =: \left\{ \left(v, q_-\right) \in \mathbb{R}_2 : \exists \sigma \in \Sigma^*\left(y_-, b\right) : \left[\begin{array}{c} v = \mathbb{E}\left\{\sum_{t=0}^{\infty} u\left(c^{\sigma}\left(h^t\right)\right)\right\} \\ c_t = y_t - b_t + q_t^{\sigma_m} b_{t+1} \\ b_0 = b \\ q_- = \frac{\mathbb{E}_{y|y_-}(1-d_0(y))}{1+r} \end{array} \right] \right\}.$$

The set $\mathcal{E}\left(y_-, b\right)$ has the (utility) values and prices that can be obtained in a SPE. Given an initial seed value y_-, recall that income follows a first order Markov process, and there are initial bonds b. In period $t = 0$, the government will repay (or not) b by choosing d_0, issuing debt b_1 at a price q_0. Next, to characterize the set of equilibrium payoffs, we will introduce a procedure that modifies the one first introduced in Abreu et al. (1990).

Step 1.1: Enforceability. Take a bounded, compact-valued correspondence $W : Y \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$.[34]

Definition 1. A government *strategy* $\left(d\left(\cdot\right), b'\left(\cdot\right)\right)$ *is enforceable* in $W\left(y_-, b\right)$ if we can find a pair of functions $v\left(\cdot\right)$ and $q\left(\cdot\right)$ such that:

a. For all $y \in Y$, $\left(v\left(y\right), q\left(y\right)\right) \in W\left(y, b'(y)\right)$

[34] We will drop the dependence on d and we will bear in mind that after default the government is not in the market. We will also interchangeably use the notation $W\left(y_-, b\right)$ and $W\left(y, b'\right)$, depending on when which one is most convenient. We find that the the notation $W\left(y, b'\right)$ is most convenient for enforceability, and the notation $W\left(y_-, b\right)$ is most convenient for the set of equilibrium payoffs.

b. For all $y \in Y$, the policy $(d(y), b'(y))$ solves the problem:

$$\max_{\hat{d}\in\{0,1\},\hat{b}\geq 0} \left(1-\hat{d}\right)\left\{u\left[y-b+q(y)\hat{b}\right]+\beta v(y)\right\}+\hat{d}\left\{u(y)+\beta\mathbb{E}_{y'|y}V^{d}(y')\right\}.$$

We refer to the pair $(v(\cdot), q(\cdot))$ as the enforcing values of policy $(d(y), b'(y))$, and we write $(d(\cdot), b'(\cdot)) \in E(W)(y_{-}, b)$. Further, given the functions $v(\cdot)$ and $q(\cdot)$ we define:

$$V^{v(\cdot),q(\cdot)}(b,y) =: \max_{\hat{d}\in\{0,1\},\hat{b}\geq 0} \left(1-\hat{d}\right)\left\{u\left[y-b+q(y)\hat{b}\right]+\beta v(y)\right\}+\hat{d}\left\{u(y)+\beta\mathbb{E}_{y'|y}V^{d}(y')\right\}.$$

Definition 2. Given a correspondence $W : Y \times \mathbb{R}_{+} \rightrightarrows \mathbb{R}^2$, we define the *generating correspondence* $B(W) : Y \times \mathbb{R}_{+} \rightrightarrows \mathbb{R}^2$ as:

$$B(W)(y,b') = \left\{(v,q)\in\mathbb{R}^2 : \exists\,(d(\cdot),b'(\cdot))\in E(W)(y,b') : \left[\begin{array}{c} v=\mathbb{E}_{y'|y}\left[V^{v(\cdot),q(\cdot)}(b',y')\right] \\ q=\frac{\mathbb{E}_{y'|y}[1-d(y)]}{1+r} \end{array}\right]\right\}.$$

The idea of $B(W)(y,b')$ is that this is the set of enforceable payoffs given the correspondence $W(\cdot,\cdot)$.

Definition 3. A correspondence $W(\cdot)$ is *self-generating* if for all $b \geq 0$ it holds that $W(y_{-}, b) \subseteq B(W)(y_{-}, b)$.

Step 1.2: A self generating set is an equilibrium set. In this step, we show that if a set of values is self-generating then it belongs to the set of equilibrium values. The proof follows Abreu et al. (1990) and is constructive; to make the manuscript as self contained as possible, we provide a brief discussion of the argument. This is now a standard argument that can be found, for the case without state variables, in different textbooks; for example Mailath and Samuelson (2006). We go back to using the notation $W(y_{-}, b)$ instead of $W(y, b')$.

Proposition 14. *Any bounded, self-generating correspondence gives equilibrium values: i.e. if $W(y_{-}, b) \subseteq B(W)(y_{-}, b)$ for all $y_{-} \in Y, b \geq 0$ then $W(y_{-}, b) \subseteq \mathcal{E}(y_{-}, b)$.*

Proof. Fix (y_{-1}, b_0). Take any pair $(v_{-1}, q_{-1}) \in W(y_{-1}, b_0)$. We would like to show that $(v_{-1}, q_{-1}) \in \mathcal{E}(y_{-1}, b_0)$. To do this, we need to construct an SPE strategy profile $\sigma \in \Sigma^{*}(y_{-1}, b_0)$ that achieves the payoff v_{-1} and in the first period generates the prices q_{-1}.[35] Next, we do just that. Since $W(y_{-1}, b_0) \subseteq B(W)(y_{-1}, b_0)$, i.e. if W is self generating, then we know we can find functions $(d_0(y_0), b_1(y_0))$, and the values $(v_0(y_0), q_0(y_0)) \in$

[35] Note that v_{-1} is the expected payoff generated by policies $\{d_t, b_{t+1}\}_{t=0}^{\infty}$ given initial bonds b_0 and the seed value for the realization of income y_{-1}; it is the ex-ante payoff from $t = 0$ onwards. In addition, q_{-1} is the price generated by thepolicy d_0.

$W(y_0, b_1(y_0))$ for any $y_0 \in Y, b_1(y_0) \geq 0$ such that:

$$(d_0(y_0), b_1(y_0)) \in \text{argmax}_{\hat{d}\in\{0,1\},\hat{b}\geq 0} \left(1-\hat{d}\right)\left\{\left[u\left(y_0 - b_0 + q(y)\hat{b}\right) + \beta v(y)\right]\right.$$

$$\left. +\hat{d}\left[u(y_0) + \beta \mathbb{E}_{y_1|y_0} V^d(y_0)\right]\right\}$$

i.e.$(d_0(y_0), b_1(y_0))$ is in the argmax of $V^{v_0(\cdot),q_0(\cdot)}(b_0, y_0)$;

$$v_{-1} = \mathbb{E}_{y_0|y_{-1}}\left\{V^{v_0(\cdot),q_0(\cdot)}(y_0, b_0)\right\};$$

$$q_{-1} = \frac{\mathbb{E}_{y_0|y_{-1}}[1 - d_0(y)]}{1+r}.$$

We define

$$\sigma_g(y_{-1}, b_0, q_{-1}) = (d_0(y_0), b_1(y_0))$$

where, for further reference, $h^0 = (y_{-1}, b_0, q_{-1})$ and

$$\sigma_m(y_{-1}, b_0, q_{-1}, y_0, d_0, b_1) = q_0$$

where $h_m^0 = (y_{-1}, b_0, q_{-1}, y_0, d_0, b_1)$. Because $(v_0(y_0), q_0(y_0)) \in W(y_0, b_1(y_0))$ and W is self-generating, we know that for any realization of y_0, we can find policy functions $(d_1(y_1), b_2(y_1))$ and values $(v_1(y_1), q_1(y_1, b_2(y_1))) \in B(W)(y_1, b_2(y_1))$ such that the policies $(d(y_1), b_2(y_1))$ are in the argmax of $V^{v_1(\cdot),q_1(\cdot)}(b_1, y_1)$ and

$$v_0(y_0) = \mathbb{E}_{y_1|y_0}\left(V^{v_1(\cdot),q_1(\cdot)}(b_1, y_1)\right),$$

$$\sigma_g\left(h^1\right) = (d_1(y_1), b_2(y_1))$$

$$\sigma_m\left(h_m^1\right) = q_1(y_1, b_2(y_1)) = \frac{\mathbb{E}_{y_1|y_0}[1 - d_1(y)]}{1+r}.$$

Note that $h^1 = (h^0, y_0, b_1, q_0)$ and $h_m^1 = (h^0, y_0, b_1, q_0, y_1, d_1, b_2)$. It is clear that the strategy profiles σ_m and σ_g thta are defined for all histories of type h^1 and h_m^1 satisfy the definition of an SPE. By doing this process recursively for all finite t, we can then prove by induction (as in Abreu et al. (1990) original's proof) that this profile forms an SPE with initial values (v_{-1}, q_{-1}), as we stated. The finiteness of the value function is guaranteed because the set W is bounded. There are no one shot deviations by construction. □

Proposition 15. *The correspondence $\mathcal{E}(y_-, b)$ is the largest correspondence (in the set order) that* *is a fixed point of B. That is,* $\mathcal{V}(\cdot)$ *satisfies:*

$$B(\mathcal{E})(y_-, b) = \mathcal{E}(y_-, b), \tag{D.1}$$

for all $y \in Y$, $b \geq 0$. *If another operator* $W(\cdot)$ *also satisfies condition D.1, then* $W(y_-, b) \subseteq \mathcal{E}(y_-, b)$ *for all* $y \in Y$, $b \geq 0$.

Proof. It is sufficient to show that $\mathcal{E}(y_-, b)$ is self-generating. As in APS, we start with any strategy profile $\sigma = (\sigma_g, \sigma_m)$, and are associated with (v_{0-1}, q_{-1}) and with initial income y_- and debt b. From the definition of SPE, we know that the policies $d_0(y_0) = d^{\sigma_g}(h^0, y_0)$ and $b'(y_0) = b_1^{\sigma_g}(h^0, y_0)$ are implementable with functions $q\left(y_0, \hat{b}\right) = q_m^{\sigma}$ $(y_0, d(y_0), b'(y_0))$ and $v\left(y_0, \hat{b}\right) = V\left(\sigma \mid h^1\left(y_0, \hat{b}\right)\right)$, where

$$h^1\left(y_0, \hat{b}\right) := (h^0, y_0, d_0(y_0), b'(y_0), q(y_0, \hat{b})).$$

Moreover, because σ is an SPE strategy profile, it is also an SPE for the continuation game starting with initial bonds $b = \hat{b}$; therefore,

$$\left(v\left(y_1, \hat{b}\right), q\left(y_1, \hat{b}\right)\right) \in \mathcal{E}\left(y_1, \hat{b}\right).$$

This then means that $(v_0, q_0) \in B(\mathcal{E})(y_-, b)$, and hence $\mathcal{E}(\cdot)$ is a self-generating correspondence. □

Step 1.4: Bang Bang Property. Next, we are going to relate the characterization in Abreu et al. (1990) with the objects introduced in the main text and in section C of the Online Appendix. First, note that the (singleton) set $\{(v, q)\} = \left\{\left(0, \mathbb{E}_{y|y_-} V^{aut}(y)\right)\right\}$, corresponding to the price and utility of autarky subgame perfect equilibria is itself self-generating and hence an equilibrium value. Second, also note that for a given (y_-, b), the values $\{(v, q)\} = \left\{\left(\overline{q}(y_-, b), \overline{\mathbb{V}}(y_-, b)\right)\right\}$ are the expected utility, and the debt price associated with the best equilibrium is also self-generating and hence an equilibrium value.

Proposition 16. *Suppose that* $(d(\cdot).b'(\cdot))$ *is an enforceable policy on* $\mathcal{E}(y_-, b)$. *This policy can be enforced by the following continuation value functions:*

$$v^{BB}\left(y, \hat{d}(y), \hat{b}'(y)\right) = \begin{cases} \overline{\mathbb{V}}(y, b'(y)) & \text{if } \hat{d}(y) = d(y) = 0 \text{ and } \hat{b}'(y) = b'(y) \\ \mathbb{E}_{y'|y} V^{aut}(y') & \text{otherwise} \end{cases} \tag{D.2}$$

and

$$q^{BB}\left(y,\hat{d}(y),\hat{b}'(y)\right)=\begin{cases}\overline{q}\left(y,b'\left(y\right)\right) & \hat{d}(y)=d\left(y\right)=0 \text{ and } \hat{b}'(y)=b'\left(y\right)\\ 0 & \textit{otherwise.}\end{cases} \tag{D.3}$$

Proof. Note that the functions $v\left(\cdot\right), q\left(\cdot\right)$ satisfy the restriction $\left(v^{BB}\left(\cdot,\cdot,\cdot\right), q^{BB}\left(\cdot,\cdot,\cdot\right)\right)\in\mathcal{E}\left(y,\hat{d}(y),\hat{b}'(y)\right)$ for all $y\in Y$. Since $\left(d\left(\cdot\right), b'\left(\cdot\right)\right)$ are enforceable, there exist functions $\left(\tilde{v}\left(\cdot\right),\tilde{q}\left(\cdot\right)\right)$ such that for all $y\in Y$ where $d\left(y\right)=0$ it holds that:

$$u\left[y-b+\tilde{q}\left(y,d(y),b'\left(y\right)\right)b'\left(y\right)\right]+\beta\tilde{v}\left(y,d(y),b'\left(y\right)\right)\geq u\left[y-b+\tilde{q}\left(y,\hat{d}(y),\hat{b}'(y)\right)\hat{b}'\left(y\right)\right] + \beta\tilde{v}\left(y,\hat{d}(y),\hat{b}'(y)\right) \tag{D.4}$$

for all $y\in Y$ and any alternative policy $\hat{d},\hat{b}'$. The left hand side of (D.4) is an equilibrium value. Thus, because it is generated by an equilibrium policy, its value must be less than the best equilibrium value for the government, characterized by $q=\overline{q}\left(y,b'\left(y\right)\right)$ and $v=\overline{\mathbb{V}}\left(y,b'\left(y\right)\right)$. Note that denotes $\overline{\mathbb{V}}\left(y,b'\left(y\right)\right)$ the best equilibrium from tomorrow on starting at a debt value of $\hat{b}=b'\left(y\right)$, for any income realization y. From these observations we know that:

$$\begin{aligned}&u\left[y-b+\overline{q}\left(y,b'\left(y\right)\right)b'\left(y\right)\right]+\beta\overline{\mathbb{V}}\left(y,b'\left(y\right)\right)\geq\\ &u\left[y-b+\tilde{q}\left(y,d(y),b'\left(y\right)\right)b'\left(y\right)\right]+\beta\tilde{v}\left(y,d(y),b'\left(y\right)\right).\end{aligned} \tag{D.5}$$

For later reference, recall that

$$V^{nd}\left(b,y,b'\left(y\right)\right)=u\left[y-b+\overline{q}\left(y,b'\left(y\right)\right)b'\left(y\right)\right]+\beta\overline{\mathbb{V}}\left(y,b'\left(y\right)\right).$$

On the other hand, it holds that that autarky is the worst equilibrium value (since it coincides with the min-max payoff). Because $\tilde{q}\left(y,\hat{d}(y),\hat{b}'(y)\right)$ and $\tilde{v}\left(y,\hat{d}(y),\hat{b}'(y)\right)$ are equilibrium values, it must be the case that:

$$u\left[y-b+\tilde{q}\left(y,\hat{d}(y),\hat{b}'(y)\right)\hat{b}'\left(y\right)\right]+\beta\tilde{v}\left(y,\hat{d}(y),\hat{b}'(y)\right)\geq u\left(y\right)+\beta\mathbb{E}_{y'|y}V^{aut}(y') \tag{D.6}$$

for all $y\in Y$. Combining (D.5) and (D.6) we obtain:

$$u\left[y-b+\overline{q}\left(y,b'\left(y\right)\right)b'\left(y\right)\right]+\beta\overline{\mathbb{V}}\left(y,b'\left(y\right)\right)\geq u\left(y\right)+\beta\mathbb{E}_{y'|y}V^{aut}(y') \tag{D.7}$$

which is the enforceability constraint (conditional on not defaulting) of the proposed functions $\left(v^{BB}, q^{BB}\right)$ in equations (D.2) and (D.3). To finish the proof, we need to show that if it is indeed optimal to choose $d\left(y\right) = 0$ under the functions $\left(\tilde{v}\left(\cdot\right), \tilde{q}\left(\cdot\right)\right)$, then it will also be so under functions $\left(v^{BB}\left(\cdot\right), q^{BB}\left(\cdot\right)\right)$. This is readily given by condition (D.7) since punishment for defaulting coincides with the value of deviating from the bond issue rule $\hat{b} = b'\left(y\right)$. Hence, $\left(v\left(\cdot\right), q\left(\cdot\right)\right)$ also enforce $\left(d\left(\cdot\right), b'\left(\cdot\right)\right)$. □

The previous proposition greatly simplifies the characterization of the implementable policies. In particular, the next corollary will be useful for the characterization in the next subsection.

Corollary 3. *A policy $\left(d\left(\cdot\right), b'\left(\cdot\right)\right)$ is enforceable on $\mathcal{E}\left(y, b'(y)\right)$ if and only if $d\left(y\right) = 0$ implies*

$$V^{nd}\left(b, y, b'\left(y\right)\right) \geq V^{d}\left(y\right).$$

Step 1.5: Monotonicity and an Iterative Procedure. One can show that $W\left(y_{-}, b\right) \subseteq W'\left(y_{-}, b\right)$ implies that $B\left(W\right)\left(y_{-}, b\right) \subseteq B\left(W'\right)\left(y_{-}, b\right)$. This is an iterative procedure used to compute the set of equilibrium payoffs that was first suggested by Abreu et al. (1990) and extended for public state variables in Atkeson (1991), Chang (1998a) and Phelan and Stacchetti (2001). In particular, starting from a compact $W_0\left(y_{-}, b\right)$ and defining $W_n\left(y_{-}, b\right) = B\left(W_{n-1}\right)\left(y_{-}, b\right)$, it holds that:

$$\mathcal{E}\left(y_{-}, b\right) = \lim_{n \to \infty} W_n\left(y_{-}, b\right).$$

Remark 1. Note that because we already characterized the best and worst equilibria, in Section C, there is no need to perform this iterative procedure for the model of sovereign debt. When the best and worst equilibria are not readily available (for example, in the general model in Section 5 of this paper), the iterative procedure, developed by Judd et al. (2003), will need to be implemented.

D.2 Step 2: Computing $\overline{v}\left(y_{-}, b, q_{-}\right)$

The function $\overline{v}\left(y_{-}, b, q_{-}\right)$ gives the highest expected utility that a government can obtain if they raised debt at price q_{-} and issued b bonds given an income realization of y_{-}. This is the Pareto frontier in the set of equilibrium values. We now discuss how we compute $\overline{v}\left(y_{-}, b, q_{-}\right)$, which can be redefined using the equilibrium correspondence:

$$\overline{v}\left(y_{-}, b, q_{-}\right) := \max\left\{v : \exists \hat{q} \geq 0 \text{ such that } \left(v, \hat{q}\right) \in \mathcal{E}\left(y_{-}, b\right) \text{ and } \hat{q} \leq q_{-}\right\} \tag{D.8}$$

Note that we focus on a relaxed version of the problem, where we replace the equality $\hat{q} = q$ by the inequality $\hat{q} \leq q$. We will show obtain a result that will enable us to compute $\overline{v}(y_-, b, q_-)$. The proof of Proposition 17 follows from the next three lemmas (1, 2, 3).

Proposition 17. *For all $q \in [0, \overline{q}(y_-, b)]$ the maximum continuation value $\overline{v}(y_-, b, q_-)$* solves

$$\overline{v}(y_-, b, q_-) = \max_{d(\cdot) \in [0,1]^Y} \mathbb{E}_{y|y_-} \left[d(y) V^d(y) + [1 - d(y)] \overline{V}^{nd}(b, y) \right]$$

subject to

$$q_- = \frac{\mathbb{E}_{y|y_-} [1 - d(y)]}{1 + r} \tag{D.9}$$

Furthermore, $\overline{v}(y_-, b, q_-)$ is non-decreasing and concave in q_-.

Lemma 1. *(Characterization of $\overline{v}$). For all $q \in [0, \overline{q}(y_-, b))$ the maximum continuation value $\overline{v}(y_-, b, q_-)$ solves*

$$\overline{v}(y_-, b, q_-) = \max_{d(\cdot) \in [0,1]^Y} \mathbb{E}_{y|y_-} \left[d(y) V^d(y) + [1 - d(y)] \overline{V}^{nd}(b, y) \right] \tag{D.10}$$

subject to

$$q_- \geq \frac{\mathbb{E}_{y|y_-} [1 - d(y)]}{1 + r}. \tag{D.11}$$

The constraint D.11 is always binding for all $q_- \geq 0$.

Proof. By definition, the ex-post best continuation value function, is given by:

$$\overline{v}(y_-, b, q_-) := \max \left\{ v : \exists \hat{q} \geq 0 \text{ such that } (v, \hat{q}) \in \mathcal{E}(y_-, b) \text{ and } \hat{q} \leq q_- \right\}.$$

Take any $\tilde{v}$ in this set; i.e.,

$$\tilde{v} \in \left\{ v : \exists \hat{q} \geq 0 \text{ such that } (v, \hat{q}) \in \mathcal{E}(y_-, b) \text{ and } \hat{q} \leq q_- \right\}.$$

Because $\tilde{v}$ is an equilibrium value, there exists an enforceable policy $\left(\tilde{d}(\cdot), \tilde{b}(\cdot)\right)$ and $\tilde{v}(y), \tilde{q}(y)$ such that:

$$\begin{aligned} \tilde{v} &= \mathbb{E}_{y|y_-} \left[\left(1 - \tilde{d}(y)\right) \left[u(y - b + \tilde{q}(y) b'(y)) + \beta \tilde{v}(y) \right] + \tilde{d}(y) V^d(y) \right] \\ \tilde{d}(y), \tilde{b}(y) &\in \arg\max_{d(y), b'(y)} (1 - d(y)) \left[u(y - b + \tilde{q}(y) b'(y)) + \beta \tilde{v}(y) \right] + d(y) V^d(y) \qquad \text{(D.12)} \\ q_- &\geq \frac{\mathbb{E}_{y|y_-} \left[1 - \tilde{d}(y)\right]}{1 + r}. \end{aligned}$$

From Proposition 16, we know that $\tilde{d}(y), \tilde{b}(y)$ is also implementable with bang bang continuation values. Therefore, we can rewrite equation (D.12) as:

$$\tilde{d}(y), \tilde{b}(y) \in \arg\max_{d(y), b'(y)} (1-d(y)) \left[u(y-b+\bar{q}(y,b'(y))b'(y)) + \beta \overline{\mathbb{V}}\left(y, b'(y)\right)\right] + d(y)V^d(y).$$

Recall that $\overline{V}^{nd}(b,y,b'(y))$ is defined as

$$\overline{V}^{nd}(b,y,b'(y)) = u(y-b+\bar{q}(y,b'(y))b'(y)) + \beta \overline{\mathbb{V}}\left(y, b'(y)\right).$$

For a given choice of $b'(y)$, it follows from Corollary (3), that the choice of $d(y)$ can be summarized as follows:

$$d(y) = 0 \iff \overline{V}^{nd}(b,y,b'(y)) \geq V^d(y).$$

Therefore, to maximize the arbitrary $\tilde{v}$, the program will now be

$$\bar{v}(y_-, b, q_-) = \max_{d(\cdot), b'(\cdot)} \quad \mathbb{E}_{y|y_-}\left[(1-d(y))\, V^{nd}\left(b, y, b'(y)\right) + d(y)V^d(y)\right]$$

subject to

$$d(y) = 0 \iff \overline{V}^{nd}(b,y,b'(y)) \geq V^d(y) \tag{D.13}$$
$$q_- \geq \frac{\mathbb{E}_{y|y_-}\left[1-\tilde{d}(y)\right]}{1+r}.$$

Note that by choosing the optimal $b'(y)$ the constraint (D.13) can be relaxed and the objective function and the value $\tilde{v}$ increase. Therefore:

$$\bar{v}(y_-, b, q_-) = \max_{d(\cdot)} \quad \mathbb{E}_{y|y_-}\left[(1-d(y))\, \overline{V}^{nd}(b,y) + d(y)V^d(y)\right]$$

subject to

$$d(y) = 0 \iff \overline{V}^{nd}(b,y) \geq V^d(y)$$
$$q_- \geq \frac{\mathbb{E}_{y|y_-}\left[1-\tilde{d}(y)\right]}{1+r}$$

Note that we can drop the constraint that characterizes $d(y) = 0$ because to maximize the function you never want to violate that constraint.

Finally, note that $\bar{v}(y_-, b, q_-)$ is *weakly increasing in* q_-. Furthermore, note that if we remove the price constraint, then the agent will choose the default rule to obtain price $\bar{q}(y_-, b)$ (the one associated with the best equilibrium), so for any $q < \bar{q}(y_-, b)$ this constraint must be binding. □

The program that characterizes $\overline{v}$ in the previous lemma is a linear programming problem in $d(\cdot)$, which as we will see is easy to solve. If tractable, this lemma will help us map the boundaries of the equilibrium correspondence $\mathcal{E}(y_-, b)$ for any given q. The following proposition solves the programming problem shown in Lemma 1, by reducing it to solving a problem of one equation in one unknown.

Lemma 2. *Given(y_-, b, q_-) there exists a constant $\gamma = \gamma(y_-, b, q_-)$ such that:*

$$\overline{v}(y_-, b, q_-) = \mathbb{E}_{y|y_-}\left[\hat{d}(y)\, V^d(y) + \left(1 - \hat{d}(y)\right)\overline{V}^{nd}(b, y)\right]$$

where

$$\hat{d}(y) = 0 \iff \overline{V}^{nd}(b, y) \geq V^d(y) + \gamma(y_-, b, q_-) \text{ for all } y \in Y$$

and γ is the (maximum) solution for the single variable equation:

$$\frac{1}{1+r}\mathbb{P}_{y|y_-}\left\{y \,:\, \overline{V}^{nd}(b, y) \geq V^d(y) + \gamma(y_-, b, q_-)\right\} = q_-.$$

Proof. From the previous lemma recall that:

$$\overline{v}(y_-, b, q_-) = \max_{d(\cdot), b'(\cdot)} \quad \mathbb{E}_{y|y_-}\left[(1 - d(y))\,\overline{V}^{nd}(b, y) + d(y)V^d(y)\right]$$

subject to

$$q_- \geq \frac{\mathbb{E}_{y|y_-}[1 - d(y)]}{1+r}.$$

Note that $d(\cdot) \in \{0, 1\}$. Following steps that are similar to the ones we followed in Proposition 5, we will solve a relaxed version of this problem in which $d(y) \in [0, 1]$. Recall that the solution will be in the corners, because we are solving a linear program. The Lagrangian is:

$$\begin{aligned}\mathcal{L} = {} & \mathbb{E}_{y|y_-}\left[(1 - d(y))\,\overline{V}^{nd}(b, y) + d(y)\, V^d(y)\right] + \\ & + \mathbb{E}_{y|y_-}\mu(y)\,[1 - d(y)]\left[\overline{V}^{nd}(b, y) - V^d(y)\right] + \\ & + \lambda\left(q(1+r) - 1 + \mathbb{E}_{y|y_-} d(y)\right).\end{aligned}$$

The first order condition with respect to $d(y)$ is given by:

$$\frac{\partial \mathcal{L}}{\partial [d(y)]} = \left[-\overline{V}^{nd}(b, y) + V^d(y) + \lambda\right] dF(y \mid y_-)$$

where $dF(y \mid y_-)$ denotes the conditional probability of state y. This implies that the optimal default rule is:

$$d(y) = \begin{cases} 0 & \text{if } \overline{V}^{nd}(b,y) \geq V^d(y) + \lambda \\ 1 & \text{otherwise} \end{cases}.$$

for every $y \in Y$, such that $\overline{V}^{nd}(b,y) \geq V^d(y)$. Defining $\gamma \equiv \lambda$ we obtain the desired result. We finally need to show that the price constraint is binding at the optimum. If this was not the case, then we could increase the objective by defaulting in fewer states of nature. □

Lemma 3. *(Concavity $\overline{v}$). The function*

$$\overline{v}(y_-, b, q_-) = \max\{v : \exists \hat{q} \leq q_- \textit{ such that } (v, \hat{q}) \in \mathcal{E}(y_-, b)\}$$

is concave in q_-.

Proof. From Lemma 1 we know that the feasible set of the program

$$\overline{v}(y_-, b, q_-) = \max\{v : \exists \hat{q} \leq q_- \text{ such that } (v, \hat{q}) \in \mathcal{E}(y_-, b)\}$$

is convex and has a linear objective function and an affine restriction. For this case $q_0, q_1 \in [0, \overline{q}(y_-, b)]$ and $\lambda \in [0,1]$. We need to show that:

$$\overline{v}(y_-, b, \lambda q_0 + (1-\lambda) q_1) \geq \lambda \overline{v}(y_-, b, q_0) + (1-\lambda)\overline{v}(y_-, b, q_1)$$

We define the functional:

$$G[d(\cdot)] =: \mathbb{E}_{y|y_-}\left[d(y)V^d(y) + [1 - d(y)]\overline{V}^{nd}(b,y)\right].$$

Note that this is the objective function of the maximization in (D.10), and the objective function of Lemma 1. Let $d_0(y)$ be one of the solutions for the program when $q = q_0$; likewise, let $d_1(y)$ be one of the solutions of the relaxed program when $q = q_1$. Define:

$$d_\lambda(y) =: \lambda d_0(y) + (1-\lambda) d_1(y).$$

Clearly, this is not a feasible default policy as it is, since d_λ may be in $(0,1)$, but it is feasible in the relaxed program of Lemma 1. Note that it is feasible when $q = q_\lambda :=$

$\lambda q_0 + (1-\lambda) q_1$, since

$$\begin{aligned}\frac{\mathbb{E}_{y|y_-}(1-d_\lambda(y))}{1+r} &= \lambda \frac{\mathbb{E}_{y|y_-}(1-d_0(y))}{1+r} + + (1-\lambda) \frac{\mathbb{E}_{y|y_-}(1-d_1(y))}{1+r} \\ &\leq \lambda q_0 + (1-\lambda) q_1 \\ &= q_\lambda .\end{aligned}$$

Therefore, the optimal continuation value at $q = q_\lambda$ must be greater than the objective function evaluated at d_λ because the optimum will be at a corner even in the relaxed problem, which implies that:

$$\begin{aligned}\overline{v}(b, q_\lambda) &\geq G[d_\lambda(\cdot)] \\ &= \lambda G[d_0(\cdot)] + (1-\lambda) G[d_1(\cdot)] \\ &= \lambda \overline{v}(y_-, b, q_0) + (1-\lambda) \overline{v}(y_-, b, q_1)\end{aligned}$$

using in the first equality the fact that $G[d(\cdot)]$ is an affine functional in $d(\cdot)$ and in the second one the fact that both $d_0(\cdot)$ and $d_1(\cdot)$ are the optimizers at q_0 and q_1 respectively. This concludes the proof of the lemma. From the three previous lemma's, we obtain the proof of the proposition. □

E Main Results of the General Model

The proofs of Propositions 7 and 8 follow the proof of Proposition 4 almost line by line,, including the preliminary results that are provided in the Online Appendix D regarding the construction of the equilibrium value correspondence. As in the first section of Online Appendix D Proposition 14, we construct the equilibrium value correspondence as the largest fixed point of the "generating values correspondence" such that for every value correspondence $\mathcal{W}(y_-, b) \subseteq \mathbb{R}^{k+1}$ i t gives a set of generating equilibrium values $B(\mathcal{W})(y_-, b) \subseteq \mathbb{R}^{k+1}$.

The first result for the general model is Proposition 7. We need continuity of the utility function and continuity, compact-valuedness and non-emptiness of the feasibility correspondence to guarantee that $\mathcal{E}(y_-, q)$ is non-empty and compact-valued, which implies that $\overline{v}(y_-, b, q)$ and $\underline{U}(b, y)$ are well-defined objects. The proof of Proposition Proposition 7 follows the proof of Proposition 4 for the case of no sunspots.

When we allow for sunspots in Proposition 8, we need to add an assumption to be able to use the same best continuation value function $\overline{v}(y_-, b, q)$ and the worst lifetime utility $\underline{U}(b, y)$. To do this, we need first that the equilibrium value correspondence must be convex-valued. This condition would be enough to make $\overline{v}(y_-, b, q)$ concave in q. However, since q enters non-linearly in the contemporaneous utility function of the long-lived player, the convexity of the equilibrium value set is not enough to guarantee that $\mathcal{E} = \mathcal{E}^s$; for this to occur, the contemporaneous utility function $u(\cdot)$ must be concave in q, as is the case in the sovereign debt model example. Armed with these two conditions (the convexity of $\mathcal{E}$ and the concavity of $u(\cdot)$) we can show the result of Proposition 8 , which which relies on the fact that $\mathcal{E} = \mathcal{E}^s$ plus the concavity of the auxiliary function $D = u + \beta\overline{v}$ to obtain the same results as those in Proposition 4. Using this proposition, $\mathcal{E} = \mathcal{E}^s$ and so are the best continuation function $\overline{v}^s = \overline{v}$ and $\underline{U}^s = \underline{U}$. We change the variable in the integration since ζ enters only through $q(y, \zeta)$. This implicitly defines a measure across prices, according to

$$\int_{\hat{q} \in \mathcal{Q}(y_t, b_{t+1})} \left[u(b_t, y_t, d_t, b_{t+1}, \hat{q}) + \beta\overline{v}(y_t, b_{t+1}, \hat{q}) \right] dQ_t(\hat{q}) \geq \underline{U}(y_t, b_t)$$

which shows the desired result.