Incentive Compatible Self-fulfilling Mechanisms and Rational Expectations*

Xuesong Huang†

April 28, 2019

Abstract

This paper extends the exact equivalence result between the outcomes of self-fulling mechanisms and rational expectations equilibrium allocations in Forges and Minelli (1997) to a large finite-agent replica economy where different replicates of the same agent are allowed to receive different private information. The first result states that the allocation realized by any incentive compatible self-fulfilling mechanism is an approximate rational expectations equilibrium allocation. Conversely, the second result states that we can associate with any given rational expectations equilibrium an incentive compatible self-fulfilling mechanism whose equilibrium allocation approximately coincides with the rational expectations equilibrium allocation.

Keywords: Rational Expectations Equilibrium, Strategic Market Game, Incentive Compatible and Self-fulfilling Mechanism, Large Economy.
JEL Codes: D04, C07

*I am deeply indebted to my advisor, Richard P. McLean, for his constant support, guidance and insightful suggestions throughout this project. I would also like to thank Oriol Carbonell and Thomas Sjöström for valuable comments.

†Department of Economics, Rutgers University, xuesong.huang@rutgers.edu

1 Introduction

Rational Expectations Equilibrium is a fundamental equilibrium concept in economics to study an economy with asymmetric information. However, there is a conceptual difficulty in understanding the informational role of prices in its framework without an explicit model of price formation.[1] Indeed, leaving the price formation unmodelled has raised several conceptual paradoxes.[2]

In a seminal paper, Forges and Minelli (1997) studied an economy with a continuum of agents where agents are classified into a finite number of types. Agents with the same type have the same initial endowment, preferences and private information. They then proposed a public communication extension of the "sell-all" market game of Shapley and Shubik (1977) that consists of two stages. In the first stage, agents report their private signals to a mediator who then publicly announces a price vector depending on the reports.[3] In the second stage, players observe this publicly announced price and play the "sell-all" market game with asymmetric information in which a player's bid for each non-monetary commodity is a function of the publicly announced price and the player's privately observed signal. The outcome of the communication game specifies market price and final allocations.[4] They introduced the notion of an incentive compatible self-fulfilling mechanism in which honestly reporting true signals in the first stage and following the mediator's "recommended" actions in the second stage is a Bayes Nash Equilibrium of the communication game.[5] Then they showed that the set of allocations realized by incentive compatible self-fulfilling mechanisms coincides with the set of rational expectations equilibrium allocations of the underlying economy.

[1]In his seminal paper on the generic existence of REE, Radner (1979) also pointed out this problem as he wrote, "A thorough theoretical analysis of this situation probably requires a more detailed specification of the trading mechanism than is usual in general equilibrium analysis."

[2]See Grossman and Stiglitz (1980), Milgrom (1981), and Dubey et al. (1987) for more detailed discussions.

[3]In this paper, we use private signal and private information interchangeably.

[4]The "sell-all" market game was first introduced by Shapley and Shubik (1977) and requires one commodity to function as money. Codognato and Ghosal (2003) proved the same equivalence result as in Forges and Minelli (1997) with a different market game called windows model which does not require the existence of money in the economy.

[5]The mechanism studied in Forges and Minelli (1997) and this paper is different from the canonical communication mechanism in the sense that in the second stage only a public signal is revealed to all agents instead of private signal for each agent. See section 4.2 for detailed discussion.

The assumptions in Forges and Minelli (1997) have two important consequences. First, the action of an individual player has no effect on the price in the market game of stage two given a continuum of players. Second, each player of a given type receives the same private information implying that the model exhibits the property of non-exclusive information. Consequently, incentive compatibility does not pose any real difficulties. It is our goal in this paper to show that the essential conclusions in Forges and Minelli (1997) are robust when we move to an economy with finitely many agents and a nontrivial information structure. We consider a sequence of replica economies with n types of consumers and r consumers of each type. As in Forges and Minelli (1997), each of the r consumers of type i has the same utility function and initial endowment. Unlike Forges and Minelli (1997), however, we allow agents with the same type to receive different private information. Now we have two technical issues to deal with. First, actions matter in the sense that, by changing his action in the strategic market game of the second stage, a player can affect the price. Second, incentive compatibility, i.e., the need to induce players to honestly report their private signals in the first stage, is a more difficult problem since non-exclusive information is no longer present. It is reasonable to conjecture that, when r is large, the influence of a single player's action on the price in the strategic market game of stage two will be small and this is indeed the case. It is also reasonable to conjecture that the influence of an individual player's reported signal on the publicly announced price in stage one will also be small if r is large. To formalize this idea, we consider sequences of economies in which the agents' signals are independent conditional on an underlying but unobserved state of nature. This allows us to bring to bear the machinery developed in McLean and Postlewaite (2004, 2005, 2017) and we are able to show that players become informationally small as r becomes large.

The main results in this paper are two approximation results for a large enough replica economy. Theorem 1 asserts that the allocations realized by any incentive compatible self-fulfilling mechanism are approximate rational expectations equilibrium allocations. Theorem 2 asserts that we can associate with any given rational expectations equilibrium an incentive compatible self-fulfilling mechanism whose equilibrium allocations approximately coincide with the rational expectations equilibrium allocations.

The remainder of this paper is organized as follows. Section 2 sets up an r-replica economy. Section 3 introduces the concept of type symmetric rational expectations equilibrium (type symmetric REE) in an r-replica economy. Section 4 discusses the communication extension of the "sell-all" market game with incomplete information and introduces the concept of incentive compatible self-fulfilling equilibrium (ICSFM) in this game. Section 5 states the main results and section 6 concludes.

2 The Environment: Replica Economy

In this paper, we will focus on a sequence of replica economies. An $r-$replica economy E^r consists of the following:

- A finite set of agents $N_r = N \times J_r$, where $N = \{1, 2, \ldots, n\}$ and $J_r = \{1, 2, \ldots, r\}$;[6]
- A finite set of commodities $\mathcal{L} = \{1, 2, \ldots, L+1\}$;
- A finite set of states of nature $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$;
- For each $i \in N$, a utility function $u_i : \mathbb{R}^{L+1}_{+} \times \Theta \to \mathbb{R}$ such that $u_{is}(\cdot,\cdot) = u_i(\cdot,\cdot)$ for all $s \in J_r$;
- For each $i \in N$, a state-independent initial endowment $w_i \in \mathbb{R}^{L+1}_{++}$ such that $w_{is} = w_i$ for all $s \in J_r$;

The state of nature is unobservable but each agent (i, s) receives a *private signal*, $t_{is} \in T_i$, which is correlated with nature's choice of θ. Here T_i is a finite set of possible private signals that agent (i, s) might receive. Note that, for agents with the same type but in different cohorts, the set of possible signals is the same. Denote $T = T_1 \times T_2 \times \cdots \times T_n$ and $T^r = T \times T \times \cdots \times T$ as the r-fold Cartesian product of r copies of T. Let $t^r = (t(1), \ldots, t(r))$ denote a generic signal

[6]In the rest of the paper, we read agent (i, s) as the *type* i agent in *cohort* s.

profile in T^r where $t(s) = (t_{1s}, t_{2s}, \ldots, t_{ns})$. If $t^r \in T^r$, we will often write $t^r = (t^r_{-is}, t_{is})$. We assume that there exists a common prior $P^r \in \Delta(\Theta \times T^r)$. We will take the point of view that P^r is the distribution of an $(rn+1)-$dimensional random vector $(\tilde{\theta}, \tilde{t}(1), \ldots, \tilde{t}(r))$ taking values in $\Theta \times T^r$ where

$$P^r(\theta, t^r) = P^r(\theta, t(1), \ldots, t(r)) = Prob\{\tilde{\theta} = \theta, \tilde{t}(1) = t(1), \ldots, \tilde{t}(r) = t(r)\}.$$

For $t^r \in T^r$, let $P^r(\cdot|t^r) \in \Delta(\Theta)$ denote the induced conditional probability measure on Θ and, for $\theta \in \Theta$, let $P^r(\cdot|\theta) \in \Delta(T^r)$ be the induced conditional probability measure on T^r. Let $I_\theta \in \Delta(\Theta)$ denote the Dirac measure that assigns probability one to state θ. We assume that $P^r(\theta, t(1), \ldots, t(n)) > 0$ for all $(\theta, t(1), \ldots, t(n)) \in \Theta \times T^r$ and that for every θ, $\hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t^r \in T^r$ such that $P^r(t^r|\theta) \neq P^r(t^r|\hat{\theta})$. Moreover, we make the following conditional independence assumption: there exists a probability measure $\lambda \in \Delta(\Theta)$ and for each $\theta \in \Theta$, each $i \in N$ there exists a probability measure $\rho_i(\cdot|\theta) \in \Delta(T_i)$ such that [7]

$$P^r(t^r|\theta) = \prod_{(i,s)\in N_r} \rho_i(t_{is}|\theta) = \prod_{s=1}^{r}\prod_{i=1}^{n} \rho_i(t_{is}|\theta),$$

and

$$P^r(t^r, \theta) = P^r(t^r|\theta)\lambda(\theta).$$

That is, conditional on the event $\tilde{\theta} = \theta$, the nr random variables $\tilde{t}_{11}, \tilde{t}_{21}, \cdots, \tilde{t}_{n1}, \cdots, \tilde{t}_{1r}, \tilde{t}_{2r}, \cdots, \tilde{t}_{nr}$ are stochastically independent. This completes our description of an r-replica economy E^r denoted as:

$$E^r = \{N_r, \mathcal{L}, \Theta, (w_i)_{(i)\in N}, (u_i)_{(i)\in N}, (T_i)_{i\in N}, P^r\}.$$

[7]This assumption is stronger than the notion of a conditionally independent sequence introduced in McLean and Postlewaite (2002) since we assume individual independence rather than cohort independence.

3 Type Symmetric Rational Expectations Equilibrium

We next formulate a notion of type symmetric rational expectations equilibrium in an r-replica economy analogous to that of Forges and Minelli (1997) for the continuum framework. We first recall some ideas in McLean and Postlewaite (2002). Given an r-replica economy E^r and $t^r \in T^r$, let $f_i(\cdot|t^r)$ denote a probability measure on T_i, the "empirical frequency distribution," defined for each $\tau_i \in T_i$ as follows:

$$f_i(\tau_i|t^r) = \frac{|\{s \in J_r | t_{is} = \tau_i\}|}{r}. \tag{1}$$

Then we define $f(t^r) = (f_1(\cdot|t^r), f_2(\cdot|t^r), \cdots, f_n(\cdot|t^r))$. Now we are ready to define a type symmetric rational expectations equilibrium in an r-replica economy.

Definition 1: A *type symmetric rational expectations equilibrium* (type symmetric REE) in an r-replica economy E^r is a pair $(q^r, (z^r_{is})_{(i,s)\in N_r})$ consisting of a price function $q^r : T^r \to \mathbb{R}^{L+1}_+$ with $q^{r,L+1}(t^r) = 1$ and an allocation function $z^r_{is} : T^r \to \mathbb{R}^{L+1}_+$ for each $(i,s) \in N_r$ satisfying:

(i) For any $t^r \in T^r$ and $\hat{t}^r \in T^r$,

$$f(t^r) = f(\hat{t}^r) \Rightarrow q^r(t^r) = q^r(\hat{t}^r). \tag{2}$$

(ii) For each $(i,s) \in N_r$, and $t^r \in T^r$,

$$z^r_{is}(t^r) \in \underset{x_i \in \beta_i(q^r(t^r))}{\operatorname{argmax}} \sum_{\theta\in\Theta} u_i(x_i,\theta) P^r(\theta | q^r(t^r), t_{is}), \tag{3}$$

where

$$\beta_i(q^r(t^r)) = \{y \in \mathbb{R}^{L+1}_+ | \sum_{l=1}^{L} q^{r,l}(t^r) y^l + y^{L+1} \le \sum_{l=1}^{L} q^{r,l}(t^r) w^l_i + w^{L+1}_i\},$$

and

$$P^r(\theta|q^r(t^r), t_{is}) = \frac{\sum\limits_{\substack{\hat{t}^r_{-is}: \\ q^r(\hat{t}^r_{-is}, t_{is}) = q^r(t^r)}} P^r(\theta, \hat{t}^r_{-is} | t_{is})}{\sum\limits_{\substack{\hat{t}^r_{-is}: \\ q^r(\hat{t}^r_{-is}, t_{is}) = q^r(t^r)}} P^r(\hat{t}^r_{-is} | t_{is})}.$$

(iii) For each $t^r \in T^r$,

$$\sum_i \sum_s z^r_{is}(t^r) = r \sum_i w_i. \tag{4}$$

(iv) For any $t^r \in T^r$, $\hat{t}^r \in T^r$, $(i,s) \in N_r$ and $(i,s') \in N_r$,

$$t_{is} = \hat{t}_{is'}, q^r(t^r) = q^r(\hat{t}^r) \Rightarrow z^r_{is}(t^r) = z^r_{is'}(\hat{t}^r). \tag{5}$$

Condition (i) says that if two signal profiles have the same empirical frequency distribution, then the corresponding type symmetric REE prices are the same. Therefore, it is the empirical frequency distribution that determines the type symmetric REE price. Condition (iii) is the market clearing condition for each $t^r \in T^r$.

According to Condition (ii), agent (i,s) chooses a bundle that maximizes expected utility conditional on the observed price $q^r(t^r)$ and his observed signal t_{is}. Finally, Condition (iv) is the type symmetry condition requiring that type i agents in different cohorts who receive the same signal and observe the same price must choose the same maximizer of the expected utility maximization problem. As a consequence of Condition (iv), we record the following:

Remark 1: For a given $t^r \in T^r$, condition (iv) implies that for any $(i,s) \in N_r$ and any $(i,s') \in N_r$,

$$t_{is} = t_{is'} \Rightarrow z^r_{is}(t^r) = z^r_{is'}(t^r). \tag{6}$$

Next, we introduce the definition of a type symmetric $\varepsilon-$ rational expectations equilibrium.

Definition 2: A *type symmetric $\varepsilon-$rational expectations equilibrium* (type symmetric $\varepsilon-$REE) in an r-replica economy E^r is a pair $(q^r, (z^r_{is})_{(i,s)\in N_r})$ consisting of a price function $q^r : T^r \to \mathbb{R}^{L+1}_+$ with $q^{r,L+1}(t^r) = 1$ and an allocation function $z^r_{is} : T^r \to \mathbb{R}^{L+1}_+$ for each $(i,s) \in N_r$ satisfying (i), (iii) and (iv) in definition 1. Furthermore, there exists a set $S^r \subseteq T^r$ such that

$$Prob\{\tilde{t}^r \in S^r\} \geq 1 - \varepsilon. \tag{7}$$

and for each $t^r \in S^r$, $(i,s) \in N_r$, and $x_i \in \beta_i(q^r(t^r))$,

$$\sum_{\theta} [u_i(z^r_{is}(t^r),\theta) - u_i(x_i,\theta)]\, P^r(\theta|q^r(t^r),t_{is}) \geq -\varepsilon. \tag{8}$$

where $\beta_i(q^r(t^r))$ and $P^r(\theta|q^r(t^r),t_{is})$ are defined as in definition 1. In words, a type symmetric $\varepsilon-$REE pair requires that, for most signal profiles t^r, the allocation $z^r_{is}(t^r)$ is almost maximizing the conditional expected utility of agent (i,s).

4 Type Symmetric Incentive Compatible Self-fulfilling Mechanisms

4.1 A Market Game with Incomplete Information

Following Forges and Minelli (1997), we associate with each r-replica economy E^r a "sell-all" market game of Shapley and Shubik (1977) with incomplete information, G_{E^r}. The game consists of $|N_r| = nr$ agents. Each agent $(i,s) \in N_r$, after receiving his private signal $t_{is} \in T_i$, chooses an action $a_{is} \in A_i$ where

$$A_i \equiv \{\left(a^1_i, ...a^L_i\right) \in \mathbb{R}^L_+ | \sum_{l=1}^{L} a^l_i \leq w^{L+1}_i\}.$$

Each action profile, $a^r = (a_{11}, a_{21}, \cdots, a_{n1}, \cdots, a_{1r}, a_{2r}, \cdots, a_{nr})$, gives rise to a price for each commodity $l \neq L+1$, defined as

$$\pi^l(a^r) = \frac{\sum_i \sum_s a^l_{is}}{r \sum_{j=1}^{n} w^l_j}.$$

The price of commodity $L+1$ is fixed at 1 irrespective of agents' actions, i.e. $\pi^{L+1}(a^r) = 1$ for each a^r. One common interpretation goes as follows: there exists a trading post for each commodity $l \in \{1, \cdots, L\}$ and commodity $L+1$ plays the role of "money". At each trading post l, each agent chooses the amount of commodity $L+1$ to bid and is required to put up for sale the entire endowment of commodity l. The final holdings of good $l \in \{1, ..., L\}$ of agent (i,s), as a function of the action profile a^r, is defined as $x_{is}[a^r] = (x^1_{is}[a^r], .., x^L_{is}[a^r], x^{L+1}_{is}[a^r])$ where

$$x^l_{is}[a^r] = \frac{a^l_{is}}{\pi^l(a^r)} \qquad \text{if } l \neq L+1;$$
$$x^{L+1}_{is}[a^r] = w^{L+1}_i + \sum_{l=1}^{L} \pi^l(a^r) w^l_i - \sum_{l=1}^{L} a^l_{is}.$$

with $x^l_{is}[a^r] = 0$ if $\pi^l(a^r) = 0$. The strategy of each agent (i, s) is a function $\sigma_{is} : T_i \to A_i$ and denote $\sigma^r = (\sigma_{11}, \sigma_{21}, \cdots, \sigma_{n1}, \cdots, \sigma_{1r}, \sigma_{2r}, \cdots, \sigma_{nr})$ and $\sigma^r(t^r) = \{\sigma^r_{is}(t_{is})\}_{(i,s)\in N_r}$. The payoff to agent (i, s) when agents choose the strategy profile σ^r is

$$v_{is}(\sigma^r) = \sum_{t^r \in T^r} \sum_{\theta \in \Theta} u_i(x_{is}[\sigma^r(t^r)], \theta) P^r(\theta, t^r).$$

where $P^r \in \Delta(\Theta \times T^r)$ is the common prior of the r-replica economy. This completes our description of the market game with incomplete information G_{E^r} denoted as:

$$G_{E^r} = \{\Theta, N_r, (\sigma_{is})_{(i,s)\in N_r}, (v_i)_{i\in N}, (T_i)_{i\in N}, P^r\}.$$

4.2 Incentive Compatible Self-fulfilling Mechanisms

We next present the communication extension of G_{E^r}, which is our finite analogue of the extension in Forges and Minelli (1997). In this communication extension, a mechanism is a mapping $\mu^r : T^r \to A^r$ and the communication game proceeds in the following way:

1. Nature chooses $\theta \in \Theta$ with probability $\lambda(\theta)$ and chooses t_{is} for each (i, s) with probability $\rho_i(t_{is}|\theta)$. Then each agent i in cohort s is informed of her private signal t_{is};

2. Every agent (i, s) submits a report $\hat{t}_{is} \in T_i$, which is not necessarily equal to t_{is}, to a mediator;

3. The mediator assembles the reported signal profile $\hat{t}^r \in T^r$ and publicly announces a price $\pi(\mu^r(\hat{t}^r))$. The function $\pi \circ \mu^r : T^r \to \mathbb{R}^{L+1}$ is assumed to be common knowledge;

4. After observing the publicly announced price $\pi(\mu^r(\hat{t}^r))$, agents play the sell-all market game with incomplete information and then the final allocations are determined.

It is important to note that this mechanism is not a canonical mechanism. That is, the mediator does not privately recommend $\mu^r_{is}(\hat{t}^r) \in A_i$ to agent (i, s) as he does in a canonical mechanism. The action $\mu^r_{is}(\hat{t}^r)$ is only "contemplated" by the mediator. All $\mu^r_{is}(\hat{t}^r)$ taken together give rise to a publicly announced price $\pi(\mu^r(\hat{t}^r))$ which is observable by all agents.[8] By incorporating a mechanism μ^r into G_{E^r}, we get an augmented game with public communication: $\Gamma_{\mu^r}(G_{E^r})$. Then in this extended game with communication, the strategy for agent (i, s) is to choose a reported signal $\hat{t}_{is} \in T_i$ and an action mapping $\delta_i : T_i \times T_i \times \mathbb{R}^{L+1} \to A_i$ that specifies a choice of an action as a function of his true signal, his reported signal and the publicly announced price. Moreover, we restrict ourselves to type symmetric mechanisms as defined below.

Definition 3: A mechanism $\mu^r : T^r \to A^r$ is *type symmetric* if for any $t^r, \hat{t}^r \in T^r$ and $ss' \in J_r$,

$$f(t^r) = f(\hat{t}^r), t_{is} = \hat{t}_{is'} \Rightarrow \mu^r_{is}(t^r) = \mu^r_{is'}(\hat{t}^r) \quad \text{for each } i \in N.$$

Remark 2: For a given $t^r \in T^r$ and a type symmetric mechanism μ^r, then for any $ss' \in J_r$,

$$t_{is} = t_{is'} \Rightarrow \mu^r_{is}(t^r) = \mu^r_{is'}(t^r). \tag{9}$$

This definition of type symmetry generalizes the one in Forges and Minelli (1997) by allowing agents with the same type but in different cohorts to have different private signals. Therefore, in a type symmetric mechanism, if $f(t^r) = f(\hat{t}^r)$ and $t_{is} = \hat{t}_{is'}$ then the mediator will contemplate the same action for agent i in the cohort s and in the cohort s'. Furthermore, the mediator must ensure that the action contemplated for (i, s) is measurable with respect to the publicly announced price and the private signal received by (i, s). This requires the mechanism to be

[8] We can certainly construct a canonical mechanism that replicates the equilibrium of the public communication device studied in this paper. However, the choice of a public communication mechanism aims to capture the idea of price being a public communication system. Admittedly, as pointed out in Forges and Minelli (1997), this choice does entail a loss of generality on the output side. **FOOTNOTE NEEDS CLARIFICATION**

adapted.

Definition 4: A mechanism $\mu^r : T^r \to A^r$ is *adapted* if for any $t^r, \hat{t}^r \in T^r$, and any agent $(i,s) \in N_r$,

$$\pi(\mu^r(t^r)) = \pi(\mu^r(\hat{t}^r)), t_{is} = \hat{t}_{is} \Rightarrow \mu^r_{is}(t^r) = \mu^r_{is}(\hat{t}^r).$$

This adaptedness condition together with the type symmetry condition correspond to condition (iv) in definition 1 of type symmetric REE. Next, we introduce the notion of incentive compatible self-fulfilling mechanism.

Definition 5: A mechanism $\mu^r : T^r \to A^r$ is a *type symmetric incentive compatible self-fulfilling mechanism* (type symmetric ICSFM) if it is type symmetric and for each $(i,s) \in N_r$, $t_{is} \in T_i$, $t'_i \in T_i$ and $\delta_i : \mathbb{R}^{L+1}_+ \to A_i$,

$$\begin{aligned} &\sum_{t^r_{-is}} \sum_{\theta} u_i(x_{is}[\mu^r_{-is}(t^r), \mu^r_{is}(t^r)], \theta) P^r(\theta, t^r_{-is} | t_{is}) \\ \geq \; &\sum_{t^r_{-is}} \sum_{\theta} u_i(x_{is}[\mu^r_{-is}(t^r_{-is}, t'_i), \delta_i(\pi(\mu^r(t^r_{-is}, t'_i))], \theta) P^r(\theta, t^r_{-is} | t_{is}). \end{aligned}$$

In other words, $\mu^r : T^r \to A^r$ is a type symmetric ICSFM if, for each agent (i,s), honestly reporting the true signal and then choosing $\mu^r_{is}(t^r) \in A_i$ yield a Bayes Nash equilibrium in $\Gamma_{\mu^r}(G_E)$.[9] In a type symmetric ICSFM, each agent chooses to tell the truth and the publicly announced price directs each agent to choose the mediator's contemplated action.

Finally, we introduce the definition of a type symmetric $\varepsilon-$incentive compatible self-fulfilling mechanism.

Definition 6: A mechanism $\mu^r : T^r \to A^r$ is a *type symmetric $\varepsilon-$incentive compatible self-fulfilling mechanism* (type symmetric $\varepsilon-$ICSFM) if the following condition is satisfied: there exists a set $S^r \subseteq T^r$ such that

$$Prob\{\tilde{t}^r \in S^r\} \geq 1 - \varepsilon \qquad (10)$$

[9] Furthermore, in this two-stage communication game every information set will be reached under any BNE and the belief is therefore uniquely determined. As a consequence, the set of BNE coincides with the set of PBE. **CHECK THIS**

and for each $t^r \in S^r$, each $(i,s) \in N_r$, $t_{is} \in T_i$, $t'_i \in T_i$ and $\delta_i : \mathbb{R}_+^{L+1} \to A_i$, we have

$$\sum_{t^r_{-is}} \sum_{\theta} u_i(x_{is}[\mu^r_{-is}(t^r), \mu^r_{is}(t^r)], \theta) P^r(\theta, t^r_{-is}|t_{is})$$
$$\geq \sum_{t^r_{-is}} \sum_{\theta} u_i(x_{is}[\mu^r_{-is}(t^r_{-is}, t'_i), \delta_i(\pi(\mu^r(t^r_{-is}, t'_i)))], \theta) P^r(\theta, t^r_{-is}|t_{is}) - \varepsilon. \tag{11}$$

In addition, we say the mechanism μ^r is a *type symmetric $\varepsilon-$restricted incentive compatible self-fulfilling mechanism* (type symmetric $\varepsilon-$RICSFM) if (11) holds only for continuous function δ_i. That is to say, if μ^r is an $\varepsilon-$ICSFM, then for most of the signal profiles, honestly reporting and following the mediator's contemplated action almost maximize an agent's payoff in $\Gamma_{\mu^r}(G_{E^r})$. If μ^r is an $\varepsilon-$RICSFM, then for most of the signal profiles, honestly reporting and following the mediator's contemplated action almost maximize an agent's payoff in $\Gamma_{\mu^r}(G_{E^r})$ with respect to continuous deviations in the second stage.

5 Main Results

In this section, we study the relationship between the set of type symmetric REE allocations in an r-replica economy and the set of allocations realized by type symmetric ICSFM in its associated communication game. Forges and Minelli (1997) showed that, in an economy with a continuum of agents, if all agents with the same type receive the same private signal, those two sets are exactly the same. In our finite-agent economy in which agents with the same type are allowed to receive different private signals, we show that for a large enough economy the following are true: the allocation realized by any type symmetric ICSFM is an approximate type symmetric REE allocation; conversely, any type symmetric REE allocation is close to the allocation associated with an type symmetric ICSFM. This section proceeds as follows: section 5.1 discusses the assumptions; section 5.2 and 5.3 state the main results and provide an informal sketch for each result. All formal proofs are in the Appendix.

5.1 Assumptions

First, we make two regularity assumptions that are similar to assumptions made in Forges and Minelli (1997).

ASSUMPTION A.1: For each $i \in N$, $u_i(\cdot, \theta)$ is continuously differentiable, strictly concave and monotonic[10]; Furthermore, $u_i(0, \theta) = 0$ and $\sum_{i \in N} w_i \gg 0$.

ASSUMPTION A.2: For each $l \neq L+1$ there exist $K^l, Q^l \in \mathbb{R}_{++}$ such that, for each $\theta \in \Theta$, $i \in N$ and $x \in \mathbb{R}_+^{L+1}$,

$$Q^l \frac{\partial u_i(x, \theta)}{\partial x^{L+1}} \leq \frac{\partial u_i(x, \theta)}{\partial x^l} \leq K^l \frac{\partial u_i(x, \theta)}{\partial x^{L+1}}. \tag{12}$$

and for each $t_i \in T_i$,

$$w_i^{L+1} \rho_i(t_i|\theta) > L \max_{l \neq L+1} [K^l \sum_{i \in N} w_i^l]. \tag{13}$$

Assumption A.1 and A.2 are conditions imposed on the replica economy to ensure that utility functions are well-behaved and that the economy has enough "money". In (12) of assumption A.2, there is a lower bound for the marginal rate of substitution which is not present in Forges and Minelli (1997). We use this assumption to show that the price determined in any type symmetric ICSFM is positive uniformly with respect to r. In particular, $\pi^l(\mu^r(t^r)) \geq Q^l > 0$ for all $l \neq L+1$ and large r. This is the analogue of the activeness assumption in page 396 of Forges and Minelli (1997), namely that the price determined in the ICSFM at each state is positive. We could have made a similar "uniform activeness" assumption to avoid the situation in which as r goes to infinity some announced price might converge to 0. Instead, we have chosen to impose our lower bound assumption on the primitives and, as shown by step 1 in the proof of Theorem 1, this lower bound assumption is enough to deduce uniform activeness.

In (13) of assumption A.2, since we are allowing agents with the same type but in different cohorts to receive different private signals, there is an extra $\rho_i(t_i|\theta)$ on the left hand side of the inequality for each t_i that does not appear in the corresponding Assumption 2 in Forges and Minelli (1997). When r is large, one can interpret $\rho_i(t_i|\theta)$ as the (approximate) fraction of i type

[10] $u_i(\cdot, \theta)$ is monotonic: if $x, y \in \mathbb{R}_+^{L+1}$, $x \geq y$ and $x \neq y$, then $u_i(x, \theta) > u_i(y, \theta)$.

agents receiving private signal t_i conditional on the event $\tilde{\theta} = \theta$. Moreover, we need one more "uniform continuity" assumption on the sequence of type symmetric REE prices for the second approximation result.

ASSUMPTION A.3: Let $\{E^r : r \geq 1\}$ be a sequence of replica economies and suppose that $\{(q^r, (z^r_{is})_{(i,s)\in N_r}) : r \geq 1\}$ is an associated sequence of type symmetric REE. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ and $\hat{r} > 0$ such that for any $r > \hat{r}$ and any $t^r, \hat{t}^r \in T^r$,

$$||f(t^r) - f(\hat{t}^r)|| < \delta \Rightarrow ||q^r(t^r) - q^r(\hat{t}^r)|| < \varepsilon. \tag{14}$$

Note that δ is only a function of ε. Recall condition (i) in definition 1 of type symmetric REE, which says that type symmetric REE prices are defined as a function of empirical frequency distributions. This assumption requires that, in a large enough economy, whenever the empirical frequency distributions of two signal profiles are close, then their corresponding type symmetric REE prices are also close.

5.2 From Incentive Compatible Self-fulfilling Mechanism to Approximate Rational Expectations Equilibrium

In this section, we present our first approximation result stating that, in a large enough replica economy, the allocation realized by any adapted type symmetric ICSFM is a type symmetric $\varepsilon-$REE allocation.

Theorem 1. *Suppose A.1 and A.2 hold. Let $\{E^r\}_{r=1}^{\infty}$ be a sequence of r-replica economies and suppose that $\{\mu^r\}_{r\geq 1}$ is a sequence of adapted, type symmetric ICSFMs. Then for every $\varepsilon > 0$, there exists an integer $\hat{r} > 0$ such that for all $r > \hat{r}$, $(q^r(\cdot), \{z^r_{is}(\cdot)\}_{(i,s)\in N_r})$ is a type symmetric $\varepsilon-$REE where for each $t^r \in T^r$*

$$q^r(t^r) = \pi(\mu^r(t^r)),$$

$$z^r_{is}(t^r) = x_{is}[\mu^r(t^r)].$$

According to this theorem, there exists a $S^r \subseteq T^r$ such that $Prob\{\tilde{t}^r \in S^r\} \approx 1$ for any large enough r and, for each $t^r \in S^r$ and agent (i,s), the allocation $x_{is}[\mu^r(t^r)]$ realized by the ICSFM μ^r is almost maximizing agent (i,s)'s expected utility given the price $\pi(\mu^r(t^r))$ determined by the market game.

In proving Theorem 1, the IC property of a type symmetric ICSFM plays no role. However, in contrast to the continuum setup of Forges and Minelli (1997), actions do "matter" in our model. That is, an action that (i,s) chooses in A_i changes the price (and therefore the allocation) in the strategic market game. However, it is reasonable to conjecture that in a large replica economy, actions will not matter too much. This is indeed the case and allows us to derive the existence of approximate type symmetric REE allocations. Having said that, our proof is complicated by the need to provide bounds for certain objects that are independent of r. The analogous bounds in Forges and Minelli (1997) are obtained more easily in the continuum framework.

We now provide an informal sketch of the argument. Fixing $\varepsilon > 0$, we apply certain machinery developed in McLean and Postlewaite (2002) (See Lemma 1 in Appendix A) to show that, for large enough r, T^r can be partitioned into $(m+1)$ subsets, $B_0^r, B_1^r, \ldots, B_m^r$ with $Prob\{\tilde{t}^r \in B_0^r\} \approx 0$. Defining $S^r = \cup_{k=1}^m B_k^r$, it follows that $Prob\{\tilde{t}^r \in S^r\} \approx 1$. To proceed, we choose a large enough r, a $t^r \in S^r$ and an agent (i,s) and then modify the argument in Forges and Minelli (1997) according to the following steps:

Step 1: We first show that $\pi^l(\mu^r(t^r)) \leq K^l$ for each $l \neq L+1$. Note that K^l is a uniform upper bound that is independent of r. Next, we show that there exists a positive constant C, also independent of r, such that $\sum_{l=1}^{L} \mu_{is}^{r,l}(t^r) \leq C < w_i^{L+1}$. The existence of such a constant C is critical for our approximation result since it avoids the situation where $\sum_{l=1}^{L} \mu_{is}^{r,l}(t^r) \to w_i^{L+1}$ as r goes to infinity. Finally, we show that $Q^l \leq \pi^l(\mu^r(t^r))$ for each $l \neq L+1$. This uniform lower bound is the analogue of the activeness assumption in Forges and Minelli (1997).

Step 2: Choose $\xi_i \in \beta_i(q^r(t^r))$. Given the uniform upper bound on $\pi(\mu^r(t^r))$ and the existence of C in step 1, we can construct a feasible action $\sigma_{is}(t^r, \alpha) \in A_i$ for agent (i, s) defined as

$$\sigma^l_{is}(t^r, \alpha) = \alpha q^l(t^r)\xi^l_i + (1-\alpha)\mu^{r,l}_{is}(t^r)$$

for $1 \leq l \leq L$ where $\alpha \in (0, 1)$ is independent of r. Next, we use the uniform lower bound on $\pi(\mu^r(t^r))$ established in step 1 to show that

$$x_{is}[\mu_{-is}(t^r), \sigma_{is}(t^r, \alpha)] \approx \alpha\xi_i + (1-\alpha)x_{is}[\mu^r(t^r)]. \tag{15}$$

for sufficiently large r. Therefore, for sufficiently large r, we have

$$\begin{aligned}
&\sum_\theta u_i(x_{is}[\mu^r(t^r)], \theta)P^r(\theta|q^r(t^r), t_{is}) \\
&\geq \sum_\theta u_i(x_{is}[\mu^r_{-is}(t^r), \sigma_{is}(t^r, \alpha)], \theta)P^r(\theta|q^r(t^r), t_{is}) \quad \text{(Definition 5 of type symmetric ICSFM)} \\
&> \sum_\theta u_i(\alpha\xi_i + (1-\alpha)x_{is}[\mu^r(t^r)], \theta)P^r(\theta|q^r(t^r), t_{is}) - \alpha\varepsilon \quad \text{(By (15))} \\
&\geq \alpha \sum_\theta u_i(\xi_i, \theta)P^r(\theta|q^r(t^r), t_{is}) + (1-\alpha)\sum_\theta u_i(x_{is}[\mu^r(t^r)], \theta)P^r(\theta|q^r(t^r), t_{is}) - \alpha\varepsilon \quad \text{(Concavity)}
\end{aligned}$$

and it follows that

$$\sum_\theta u_i(x_{is}[\mu^r(t^r)], \theta)P^r(\theta|q^r(t^r), t_{is}) \geq \sum_\theta u_i(\xi_i, \theta)P^r(\theta|q^r(t^r), t_{is}) - \varepsilon.$$

5.3 From Rational Expectations Equilibrium to Approximate Incentive Compatible Self-fulfilling Mechanism

Now we state our second result which asserts that, in a large enough replica economy, we can associate with any given type symmetric REE a type symmetric $\varepsilon-$ICSFM whose equilibrium allocation coincides with the type symmetric REE allocation, except for the "money" commodity.

Theorem 2. *Suppose A.1 and A.2 hold. Let $\{E^r\}^\infty_{r=1}$ be a sequence of replica economies and* *suppose* $\{(q^r, (z^r_{is})_{(i,s)\in N_r})\}_{r\geq 1}$ *is a sequence of type symmetric REE. Furthermore, suppose the sequence* $\{q^r\}_{r\geq 1}$ *of the type symmetric REE prices satisfies A.3. Then for every* $\varepsilon > 0$*, there exists an integer* $\hat{r} > 0$ *such that for all* $r > \hat{r}$ *there exists a type symmetric* $\varepsilon-$*ICSFM* $\mu^r(\cdot)$ *such that for any* $(i,s) \in N_r$ *and any* $t^r \in T^r$,

$$||\pi(\mu^r(t^r)) - q^r(t^r)|| < \varepsilon, \tag{16}$$

$$x^l_{is}[\mu^r(t^r)] = z^{r,l}_{is}(t^r) \text{ for all } l \neq L+1 \ . \tag{17}$$

$$|x^{L+1}_{i,s}[\mu^r(t^r)] - z^{r,L+1}_{i,s}(t^r)| < \varepsilon. \tag{18}$$

To understand our result, let us first recall the result in Forges and Minelli (1997). With a continuum of agents and non-exclusive information, Forges and Minelli (1997) showed that for any given type symmetric REE there exists a type symmetric self-fulfilling mechanism whose equilibrium allocation and price coincide with the type symmetric REE allocation and price. Our theorem 2, however, shows that in our finite-agent setup with a nontrivial information structure, for any given type symmetric REE, there exists an approximate type symmetric *incentive compatible* self-fulfilling mechanism whose equilibrium allocation coincides with the type symmetric REE allocation for all commodities except the $L+1$ commodity, i.e., the "money." The equilibrium price realized by the approximate type symmetric ICSFM is close to the type symmetric REE price, which renders the approximate equivalence of the "money" commodity.

The proof uses the machinery of informational size formally developed in McLean and Postlewaite (2002) to address the issue of incentive compatibility that is absent in Forges and Minelli (1997). If an agent's informational size is small, then the profitability of misreporting is also small. As shown in McLean and Postlewaite (2002), each agent's informational size can be made arbitrarily small for a large enough replica economy. Therefore, we can construct a mechanism μ^r in which honest reporting and following the contemplated action is almost maximizing each agent's payoff.

Now we provide an informal sketch of the argument. Fixing $\varepsilon > 0$, similar to the proof of Theorem 1, we apply certain machinery developed in McLean and Postlewaite (2002) (See Lemma 1 in Appendix A) to show that, for a large enough r, T^r can be partitioned into $(m+1)$ subsets, $B_0^r, B_1^r, \cdots, B_m^r$ with the following properties: 1) $Prob\{\tilde{t}^r \in B_0^r\} \approx 0$; 2) For each $k \geq 1$, if $t^r, \hat{t}^r \in B_k^r$, then $f(t^r) \approx f(\hat{t}^r)$. Defining $S^r = \cup_{k=1}^m B_k^r$, it follows that $Prob\{\tilde{t}^r \in S^r\} \approx 1$. Choosing a large enough r, we proceed in the following steps:

Step 1: As in the sketch of the proof of Theorem 1, we begin by showing that there exist bounds K^l, Q^l, independent of r, such that for each $l \neq L+1$ and each $t^r \in T^r$,

$$Q^l \leq q^r(t^r) \leq K^l.$$

Step 2: Construct the mechanism $\mu^r : T^r \to A^r$ in the following way: first, for each $k \in J_m$ choose a $\bar{t}^{r,k} \in B_k^r$ and let $\bar{q}_k^r = q^r(\bar{t}^{r,k})$. Second, for each (i,s) and each $l \neq L+1$, let

$$\mu_{i,s}^{r,l}(t^r) = \begin{cases} \bar{q}_k^{r,l} z_{i,s}^{r,l}(t^r) & \text{if } t^r \in B_k^r \\ q^{r,l}(t^r) z_{i,s}^{r,l}(t^r) & \text{if } t^r \in B_0^r. \end{cases}$$

Note first that $\pi(\mu(t^r)) = \bar{q}_k^r$ for all $t^r \in B_k^r$. Then by property 2) of the partition and assumption A.3, for any $t^r \in B_k^r$ it follows that $f(t^r) \approx f(\bar{t}^{r,k})$. Consequently, $\pi(\mu(t^r)) = \bar{q}_k^r \approx q^r(t^r)$, implying that $x_{is}[\mu^r(t^r)] \approx z_{is}^r(t^r)$. Similar to Forges and Minelli (1997), we use the uniform upper bound on the type symmetric REE prices established in step 1 to show that the constructed mechanism μ^r is feasible for all $t^r \in S^r$.

Step 3: To show the construct mechanism μ^r satisfies (11) in the definition of ε-ICSFM, we choose an agent (i,s), her signal $t_i \in T_i$, a $t_i' \in T_i$ and $\delta_i : \mathbb{R}^{L+1} \to A_i$. Then we break agent (i,s)'s deviation into two parts: the first part isolates the effect of misreporting and the second part isolates the effect of choosing a different action.

Step 3.1: To isolate the effect of misreporting, consider

$$M_1 = \sum_{t^r_{-is}} \sum_{\theta} u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\pi(\mu(t^r_{-is}, t_i)))], \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t'_i), \delta_i(\pi(\mu(t^r_{-is}, t'_i)))], \theta)] P^r(\theta, t^r_{-is}|t_i)$$

Since $Prob\{\tilde{t}^r \in B^r_0\} \approx 0$ and $Prob\{(\tilde{t}_{-is}, t_i) \in B^r_k$ and $(\tilde{t}_{-is}, t'_i) \in B^r_0\} \approx 0$, it follows that[11]

$$\begin{aligned} M_1 &\geq \sum_k \sum_{\substack{t^r_{-is} \\ :(t^r_{-is}, t_i) \in B^r_k \\ (t^r_{-is}, t'_i) \in B^r_k}} \sum_{\theta} [u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\bar{q}^r_k)], \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t'_i), \delta_i(\bar{q}^r_k)], \theta)] P^r(\theta, t^r_{-is}|t_i) - \frac{\varepsilon}{4} \\ &\geq -\frac{\varepsilon}{2}. \end{aligned}$$

The first inequality follows from small informational size. To justify the second inequality, note that $(t^r_{-is}, t_i) \in B^r_k$ and $(t^r_{-is}, t'_i) \in B^r_k$, then the construction of the mechanism implies that $\pi(\mu^r(t^r_{-is}, t_i)) = \bar{q}^r_k = \pi(\mu^r(t^r_{-is}, t'_i))$. Therefore, we have

$$x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\bar{q}^r_k)] \approx x_{is}[\mu_{-is}(t^r_{-is}, t'_i), \delta_i(\bar{q}^r_k)].$$

Step 3.2: First, define $Q(t_i) = \{q^r(t^r_{-is}, t_i) | (t^r_{-is}, t_i) \in T^r \backslash B^r_0\}$. By the condition (ii) in definition 1 of type symmetric REE, there exists a function $\hat{z}_{is}(q, t_i)$ such that $z^r_{is}(t^r_{-is}, t_i) = \hat{z}_{is}(q, t_i)$ for all t^r_{-is} satisfying $q^r(t^r_{-is}, t_i) = q$. Next, for any $a_i \in A_i$ and $q \in \mathbb{R}^L_{++}$ define

$$y_{is}[a_i|q] \equiv (\frac{a^1_i}{q^1}, \cdots, \frac{a^L_i}{q^L}, w^{L+1}_i + \sum_{l=1}^{L} q^l w^l_i - \sum_{l=1}^{L} a^l_i).$$

In words, $y_{is}[a_i|q]$ is the allocation for agent (i, s) if he chooses action a_i and the market game price is exogenously given as q. Note that $y_{is}[a_i|q] \in \beta_i(q)$ by definition. Furthermore, for a large enough r, we can use the uniform lower bound on the type symmetric REE prices established in

[11] See step 2 in the proof of Theorem 2 in Appendix C.

step 1 to show that $x_{is}[\mu^r_{-is}(t^r), a_i] \approx y_{is}[a_i|q^r(t^r)]$. Now we consider the second part:

$$\sum_{t^r_{-is}}\sum_{\theta}[u_i(x_{is}[\mu(t^r_{-is}, t_i)], \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\pi(\mu(t^r_{-is}, t'_i)))], \theta)]P^r(\theta, t^r_{-is}|t_i)$$

$$(\ x_{is}[\mu(t^r_{-is}, t_i)] \approx z_{is}(t^r_{-is}, t_i).\ )$$

$$\geq \sum_{t^r_{-is}}\sum_{\theta}[u_i(z_{is}(t^r_{-is}, t_i), \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\pi(\mu(t^r_{-is}, t'_i)))], \theta)]P^r(\theta, t^r_{-is}|t_i) - \frac{\varepsilon}{16}$$

$$(\ Prob\{\tilde{t}^r \in B^r_0\} \approx 0 \text{ and } z^r_{is}(t^r_{-is}, t_i) = \hat{z}_{is}(q, t_i).\ )$$

$$\geq \sum_{q\in Q(t_i)}\sum_{k}\sum_{\substack{t^r_{-is}:\\ (t^r_{-is},t_i)\in B^r_k\\ q(t^r_{-is},t_i)=q}}\sum_{\theta}[u_i(\hat{z}_{is}(q, t_i), \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\bar{q}^r_k)], \theta)]P^r(\theta, t^r_{-is}|t_i) - \frac{\varepsilon}{8}$$

$$(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\bar{q}^r_k)] \approx y_{i,s}[\delta_i(\bar{q}^r_k)|q].)$$

$$\geq \sum_{q\in Q(t_i)}\sum_{k}\sum_{\substack{t^r_{-is}:\\ (t^r_{-is},t_i)\in B^r_k\\ q(t^r_{-is},t_i)=q}}\sum_{\theta}[u_i(\hat{z}_{is}(q, t_i), \theta) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q], \theta)]P^r(\theta, t^r_{-is}|t_i) - \frac{\varepsilon}{4} \tag{19}$$

(By Step 2 in the proof of Theorem 2 in Appendix C.)

$$\geq \sum_{q\in Q(t_i)}\sum_{k}[u_i(\hat{z}_{is}(q, t_i), \theta_k) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q], \theta_k)]\frac{\sum\limits_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(\theta_k, t^r_{-is}|t_i)}{\sum\limits_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i)}\sum_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i) - \frac{\varepsilon}{2} \tag{20}$$

$$= \sum_{q\in Q(t_i)}\{\sum_{k}[u_i(\hat{z}_{is}(q, t_i), \theta_k) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q], \theta_k)]P^r(\theta_k|q, t_i)\}\sum_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i) - \frac{\varepsilon}{2}$$

(By definition 1 of type symmetric REE and $x_{is}[\delta_i(\bar{q}^r_k)|q] \in \beta_i(q)$)

$$\geq -\frac{\varepsilon}{2}.$$

Remark 4: Our construction of the mechanism in Theorem 2 is different from the more "natural" construction in Forges and Minelli (1997) which is $\mu^{r,l}(t^r) = q^{r,l}(t^r)z^{r,l}_{is}(t^r)$ for each (i, s), each $l \neq L+1$ and each $t^r \in T^r$. Formally, this is due to the critical step from (19) to (20) where we want to move the summation over t^r_{-is} with respect to q to be in front of $P^r(\theta, t^r_{-is}|t_i)$. If we constructed the mechanism as in Forges and Minelli (1997), then $y_{is}[\delta_i(\pi(\mu^r(t^r)))|q] = y_{is}[\delta_i(q^r(t^r_{-is}, t_i))|q]$ would depend on the choice t^r_{-is} and therefore the logic from (19) to (20) would no longer hold. We can however make the construction in Forges and Minelli (1997) work, if we restrict ourselves to type symmetric $\varepsilon-$RICSFM. In this case, the continuity of δ_i implies that $y_{is}[\delta_i(\pi(\mu^r(t^r)))|q] = y_{is}[\delta_i(q^r(t^r_{-is}, t_i))|q] \approx y_{is}[\delta_i(\bar{q}^r_k)|q]$. Now we state our Theorem 3.

Theorem 3. *Suppose A.1 and A.2 hold. Let* $\{E^r\}_{r=1}^{\infty}$ *be a sequence of replica economies and suppose* $\{(q^r, (z^r_{is})_{(i,s)\in N_r})\}_{r\geq 1}$ *is a sequence of type symmetric REE. Furthermore, suppose the sequence* $\{q^r\}_{r\geq 1}$ *of the type symmetric REE prices satisfies A.3. Then for every* $\varepsilon > 0$*, there exists an integer* $\hat{r} > 0$ *such that for all* $r > \hat{r}$ *there exists an* $\varepsilon-$*TSRICSFM* $\mu^r(\cdot)$ *such that for any* $(i, s) \in N_r$ *and any* $t^r \in T^r$*,*

$$\pi(\mu^r(t^r)) = q^r(t^r), \tag{21}$$

$$x_{is}[\mu^r(t^r)] = z^r_{is}(t^r)\ . \tag{22}$$

6 Discussion

1. In this paper, we follow Forges and Minelli (1997) by using the so-called "sell-all" market game to model the price formation. There are several other strategic market games that are more complicated and perhaps more realistic than the "sell-all" market game, such as the one studied in Postlewaite and Schmeidler (1978). However, the main focus of this paper is to carefully address the issue of incentive compatibility which arises naturally in our setup. It is certainly worthwhile to check the robustness of our results for other formulations of strategic market game.

2. In this paper, we use the strong Law of Large Numbers to obtain the approximate equivalence result. One interesting extension is to use the exact Law of Large Numbers introduced by Sun (2006) to obtain an exact equivalence result with a continuum of agents and with more general

information structure like the one studied in this paper.

3. Regarding the interpretation of the mediator in our mechanism, it is not reasonable to believe that there does exist a central authority that functions in the way we described in this paper. However, Forges and Minelli (1997) showed that the self-fulfilling mechanism can be replicated by a stationary nash equilibrium in an infinitely repeated market game. Therefore, the "mediator" does not need to be an actual central authority. The extension of their results in our setup is an interesting question for future research.

Appendix

Appendix A: Preliminary Lemmas

First, given a replica economy E^r and $\varepsilon > 0$, we define the partition $\Pi^r(\varepsilon) = \{B_0^r(\varepsilon), B_1^r(\varepsilon), \ldots, B_m^r(\varepsilon)\}$ of T^r as the following:[12]

$$B_k^r(\varepsilon) = \{t^r \in T^r : ||P^r(\cdot|t^r) - I_{\theta_k}|| \leq \varepsilon\} \quad \text{for all } k \in J_m,$$

and

$$B_0^r(\varepsilon) = T^r \setminus (\cup_{k \in J_m} B_k^r(\varepsilon)),$$

where $J_m = \{1, 2, \ldots, m\}$.

Lemma 1(McLean and Postlewaite (2002)): Let $\{E^r\}_{r=1}^{\infty}$ be a sequence of replica economies. For every $\varepsilon > 0$, there exists an $\hat{r}$ such that for any $r > \hat{r}$ and associated partition $\Pi^r(\varepsilon) = \{B_0^r(\varepsilon), \ldots, B_m^r(\varepsilon)\}$ of T^r the following hold:

i)

$$Prob\{\tilde{t}^r \in \cup_{k \in J_m} B_k^r(\varepsilon)\} \geq 1 - \varepsilon. \tag{23}$$

ii) For each (i, s), each t_{is} and each t_i',

$$\sum_{k=1}^{m} \sum_{\substack{t_{-is}^r: \\ (t_{-is}^r, t_{is}) \in B_k^r(\varepsilon) \\ (t_{-is}^r, t_i') \notin B_k^r(\varepsilon) \cup B_0^r(\varepsilon)}} P^r(t_{-is}^r | t_{is}) \leq \varepsilon, \tag{24}$$

and

$$\sum_{\substack{t_{-is}^r: \\ (t_{-is}^r, t_{is}) \in B_0^r(\varepsilon)}} P^r(t_{-is}^r | t_{is}) \leq \varepsilon. \tag{25}$$

iii) For each (i, s) and each t_{is},

$$\sum_k |P^r(\theta_k | t_{is}) - Prob\{\tilde{t}^r \in B_k^r(\varepsilon) | \tilde{t}_{is} = t_{is}\}| \leq \varepsilon. \tag{26}$$

iv) For each k, each $t^r \in B_k^r(\varepsilon)$ and each i,

$$\sum_{t_i \in T_i} |f_i(t_i | t^r) - \rho_i(t_i | \theta_k)| < \varepsilon. \tag{27}$$

Lemma 2: Suppose $\{x_r\}_{r=1}^{\infty}$ and $\{y_r\}_{r=1}^{\infty}$ are sequences in $\mathbb{R}^1$. Suppose that $\lim_{r \to \infty} (x_r - y_r) = 0$ and $y_r \geq A > 0$ for all r. Then $\lim_{r \to \infty} (\frac{1}{x_r} - \frac{1}{y_r}) = 0$.

Proof. Choose $\varepsilon > 0$. First note that there exists an $\hat{r}_1 > 0$ such that for any $r > \hat{r}_1$, $x_r > y_r - \frac{A}{2} \geq \frac{A}{2}$. Moreover, there exists an $\hat{r}_2$ such that for any $r > \hat{r}_2$, $|x_r - y_r| < \frac{A^2 \varepsilon}{2}$. Then, it follows that there exists an $\hat{r} = \max\{\hat{r}_1, \hat{r}_2\}$ such that for any $r > \hat{r}$,

$$|\frac{1}{x_r} - \frac{1}{y_r}| = |\frac{y_r - x_r}{x_r y_r}| < \varepsilon.$$

[12] Here, we need $\varepsilon < \frac{1}{\sqrt{2}}$ so that $\{B_k^r(\varepsilon) : k \in J_m\}$ are disjoint subsets of T^r. For the rest of the proof, we assume $\varepsilon < \frac{1}{\sqrt{2}}$.

That is, $\lim_{r\to\infty}(\frac{1}{x_r}-\frac{1}{y_r})=0$. ■

Appendix B: Proof of Theorem 1

From now on, to simplify notations, we will write $\mu(t^r)$ for $\mu^r(t^r)$, $q(t^r)$ for $q^r(t^r)$ and $z_{i,s}(t^r)$ for $z^r_{i,s}(t^r)$.
Step 0: First, note that by (13) in assumption A.2 there exists a positive number C such that for each θ, each $i\in N$ and each $t_i\in T_i$,

$$w_i^{L+1} > C > \frac{L\max_{l\neq L+1}[K^l\sum_{i\in N} w_i^l]}{\rho_i(t_i|\theta)}.$$

Choose $0<\eta^*<\rho_i(t_i|\theta)$ so that for each θ, each $i\in N$ and each $t_i\in T_i$,

$$C > \frac{L\max_{l\neq L+1}[K^l\sum_{i\in N} w_i^l]}{\rho_i(t_i|\theta)-\eta}. \tag{28}$$

Step 1: For any $\eta\in(0,\eta^*)$, there exists an $\hat{r}>0$ such that for any $r>\hat{r}$, any $k\in J_m$, any $t^r\in B_k^r(\eta)$ and any $l\neq L+1$,

$$Q^l\leq \pi^l(\mu(t^r))\leq K^l.$$

To begin, first choose $\eta\in(0,\eta^*)$ and then we proceed in the following steps.

Step 1.1: There exists an $\hat{r}_1>0$ such that for any $r>\hat{r}_1$, any $l\neq L+1$ and any $t^r\in T^r$,

$$\pi^l(\mu(t^r))\leq K^l.$$

To prove this by contradiction, first suppose that for any $\hat{r}_1>0$ there exists an $r>\hat{r}_1$, a $\hat{l}\neq L+1$ and a $t^r\in T^r$ such that

$$K^{\hat{l}} < \pi^{\hat{l}}(\mu(t^r)). \tag{29}$$

Then choose $\hat{r}_1>0$ such that there exists an $r>\hat{r}_1$, a $\hat{l}\neq L+1$ and a $t^r\in T^r$ such that

$$\pi^{\hat{l}}(\mu(t^r)) - \frac{w_i^{L+1}}{r\sum_{i=1}^n W_i^{\hat{l}}} > \frac{K^{\hat{l}}}{2}. \tag{30}$$

Applying assumption A.2, it follows that for agent (i,s)

$$\begin{aligned}
\frac{\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{\hat{l}}}P^r(\theta|\pi(\mu(t^r)),t_{is})}{\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}}P^r(\theta|\pi(\mu(t^r)),t_{is})} &= \sum_\theta\left[\frac{\frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{\hat{l}}}P^r(\theta|\pi(\mu(t^r)),t_{is})}{\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}}P^r(\theta|\pi(\mu(t^r)),t_{is})}\right]\\
&= \sum_\theta\left[\frac{\frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{\hat{l}}}P^r(\theta|\pi(\mu(t^r)),t_{is})}{\frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}}P^r(\theta|\pi(\mu(t^r)),t_{is})}\right]\left[\frac{\frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}}P^r(\theta|\pi(\mu(t^r)),t_{is})}{\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}}P^r(\theta|\pi(\mu(t^r)),t_{is})}\right]\\
&\leq K^{\hat{l}} \quad \text{(By (12))}.
\end{aligned}$$

Together with (29), we obtain

$$\frac{\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{\hat{l}}}P^r(\theta|\pi(\mu(t^r)),t_{is})}{\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}}P^r(\theta|\pi(\mu(t^r)),t_{is})} \leq K^{\hat{l}} < \pi^{\hat{l}}(\mu(t^r)). \tag{31}$$

The definition of $\pi^{\hat{l}}(\mu(t^r))$ implies that there exists an agent $(i,s) \in N_r$ such that $\mu_{is}^{\hat{l}}(t^r) \geq \pi^{\hat{l}}(\mu(t^r))w_i^{\hat{l}}$. Since $\pi^{\hat{l}}(\mu(t^r)) > K^{\hat{l}} > 0$, it follows that $\mu_{is}^{\hat{l}}(t^r) \geq \pi^{\hat{l}}(\mu(t^r))w_i^{\hat{l}} > K^{\hat{l}}w_i^{\hat{l}} > 0$. For $\gamma \leq \mu_{is}^{\hat{l}}(t^r)$, consider $a_{is}(\gamma) \in A_i$ where

$$a_{is}(\gamma) = (\mu_{is}^1(t^r), \cdots, \mu_{is}^{\hat{l}}(t^r) - \gamma, \cdots, \mu_{is}^L(t^r)).$$

Its corresponding allocation is

$$\begin{aligned}
x_{is}^l[\mu_{-is}(t^r), a_{is}(\gamma)] &= x_{is}^l[\mu(t^r)] \quad \text{ for any } l \neq \hat{l};\\
x_{is}^{\hat{l}}[\mu_{-is}(t^r), a_{is}(\gamma)] &= \frac{\mu_{is}^{\hat{l}}(t^r) - \gamma}{\pi^{\hat{l}}(\mu(t^r)) - \frac{\gamma}{r\sum\limits_i w_i^{\hat{l}}}};\\
x_{is}^{L+1}[\mu_{-is}(t^r), a_{is}(\gamma)] &= x_{is}^{L+1}[\mu(t^r)] + \gamma(1 - \frac{w_i^{\hat{l}}}{r\sum\limits_j w_j^{\hat{l}}}).
\end{aligned}$$

Let $x_{is}(r,\gamma) = x_{is}[\mu_{-is}(t^r), a_{is}(\gamma)]$ and $v_i(r,\gamma) = \sum\limits_\theta u_i(x_{is}(r,\gamma),\theta)P^r(\theta|\pi(\mu(t^r)), t_{is})$. Note that $x_{is}(r,0) = x_{is}[\mu(t^r)]$ and $v_i(r,0) = \sum\limits_\theta u_i(x_{is}[\mu(t^r)],\theta)P^r(\theta|\pi(\mu(t^r)), t_{is})$. We will show that γ can be chosen so that $a_{is}(\gamma)$ is an element of A_i and $v_i(r,\gamma) > v_i(r,0)$. This contradiction will complete the proof of step 1.1.
Next, define

$$g(\delta) = \frac{\delta\pi^{\hat{l}}(\mu(t^r))}{\frac{\delta}{r\sum\limits_i w_i^{\hat{l}}} + [\pi^{\hat{l}}(\mu(t^r)) - \frac{\mu_{is}^{\hat{l}}(t^r)}{r\sum\limits_i w_i^{\hat{l}}}]}.$$

for each $\delta \geq 0$. Note that

$$\begin{aligned}
x_{is}^l[\mu_{-is}(t^r), a_{is}(g(\delta))] &= x_{is}^l[\mu(t^r)] \quad \text{ for any } l \neq \hat{l};\\
x_{is}^{\hat{l}}[\mu_{-is}(t^r), a_{is}(g(\delta))] &= \frac{\mu_{is}^{\hat{l}}(t^r) - \delta}{\pi^{\hat{l}}(\mu(t^r))};\\
x_{is}^{L+1}[\mu_{-is}(t^r), a_{is}(g(\delta))] &= x_{is}^{L+1}[\mu(t^r)] + g(\delta)(1 - \frac{w_i^{\hat{l}}}{r\sum\limits_j w_j^{\hat{l}}}).
\end{aligned}$$

By (30), $\pi^{\hat{l}}(\mu(t^r)) - \frac{\mu_{is}^{\hat{l}}(t^r)}{r\sum\limits_i w_i^{\hat{l}}}$ is positive. Therefore, $g(\delta) > 0$ for all $\delta > 0$ and $g(0) = 0$. Furthermore, g is differentiable at $\delta = 0$ with

$$g'(0) = \frac{\pi^{\hat{l}}(\mu(t^r))}{\pi^{\hat{l}}(\mu(t^r)) - \frac{\mu_{is}^{\hat{l}}(t^r)}{r\sum\limits_i w_i^{\hat{l}}}}.$$

Therefore, the derivative of $v_i(r, g(\delta))$ with respect to δ is

$$\frac{\partial v_i(r,g(\delta))}{\partial\delta} = \sum_\theta \left[-\frac{\partial u_i(x_{is}(r,\delta),\theta)}{\partial x^{\hat{l}}} \frac{1}{\pi^{\hat{l}}(\mu(t^r))} + \frac{\partial u_i(x_{is}(r,\delta),\theta)}{\partial x^{L+1}} g'(\delta)(1 - \frac{w_i^{\hat{l}}}{r\sum\limits_j w_j^{\hat{l}}}) \right] P^r(\theta|\pi(\mu(t^r)), t_{is}).$$

Note that $\mu_{is}^{\hat{l}}(t^r) \geq \pi^{\hat{l}}(\mu(t^r))w_i^{\hat{l}}$ implies that

$$g'(0)(1 - \frac{w_i^{\hat{l}}}{r\sum\limits_j w_j^{\hat{l}}}) = \frac{\pi^{\hat{l}}(\mu(t^r))r\sum\limits_j w_j^{\hat{l}} - \pi^{\hat{l}}(\mu(t^r))w_i^{\hat{l}}}{\pi^{\hat{l}}(\mu(t^r))r\sum\limits_j w_j^{\hat{l}} - \mu_{is}^{\hat{l}}(t^r)} \geq 1.$$

Together with (31), at $\delta = 0$ we have

$$\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{\hat{l}}} P^r(\theta|\pi(\mu(t^r)), t_{is}) \frac{1}{\pi^{\hat{l}}(\mu(t^r))} < \sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}} P^r(\theta|\pi(\mu(t^r)), t_{is})$$
$$\leq \sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}} g'(0)(1 - \frac{w_i^{\hat{l}}}{r\sum_j w_j^{\hat{l}}}) P^r(\theta|\pi(\mu(t^r)), t_{is}).$$

Therefore $\frac{\partial v_i(r,g(\delta))}{\partial \delta}|_{\delta=0} > 0$ implies that there exists a $\delta^* > 0$ such that for any $0 < \delta < \delta^*$,

$$v_i(r, g(\delta)) > v_i(r, g(0)) = v_i(r, 0).$$

By (30), we have $\lim_{\delta\to 0} g(\delta) = 0$. Therefore, choose $0 < \delta < \delta^*$ such that $\mu_{is}^{\hat{l}}(t^r) - g(\delta) > 0$. It follows that $a_{is}(g(\delta)) \in A_i$ is feasible for agent (i,s) and $v_i(r, g(\delta)) > v_i(r, 0)$. This contradicts the assumption that μ is an ICSFM.

Step 1.2: There exists an $\hat{r}_2 > 0$ such that for any $r > \hat{r}_2$, any $k \in J_m$, any $t^r \in B_k^r(\eta)$ and any $(i,s) \in N_r$,

$$\sum_{l=1}^{L} \mu_{is}^l(t^r) \leq C < w_i^{L+1}. \tag{32}$$

To prove (32) by contradiction, suppose that for any $\hat{r}_2 > 0$, there exists an $r > \hat{r}_2$, an agent $(i,s) \in N_r$, a $k \in J_m$ and a type profile $t^r \in B_k^r(\eta)$ such that

$$\sum_{l=1}^{L} \mu_{is}^l(t^r) > C.$$

It follows that there exists an $\hat{l}$ such that

$$\mu_{is}^{\hat{l}}(t^r) > \frac{C}{L}. \tag{33}$$

Applying step 1.1 and Lemma 1.iv), we choose $\hat{r}_2$ such that there exists an $r > \hat{r}_2$, an agent $(i,s) \in N_r$, a $k \in J_m$, a type profile $t^r \in B_k^r(\eta)$ and an commodity $\hat{l}$ satisfying (33), $K^{\hat{l}} \geq \pi^{\hat{l}}(\mu(t^r))$ and

$$f_i(t_i|t^r) > \rho_i(t_i|\theta_k) - \eta. \tag{34}$$

If $t_{is}^r = t_i$, then it follows that

$$\begin{aligned}
K^{\hat{l}} &\geq \pi^{\hat{l}}(\mu(t^r)) \quad \text{(Step 1.1)} \\
&= \frac{\sum_i\sum_s \mu_{is}^{\hat{l}}(t^r)}{r\sum_i w_i^{\hat{l}}} = \frac{\sum_s \mu_{is}^{\hat{l}}(t^r)}{r\sum_i w_i^{\hat{l}}} + \frac{\sum_{j\neq i}\sum_s \mu_{js}^{\hat{l}}(t^r)}{r\sum_i w_i^{\hat{l}}} \\
&\geq \frac{\sum_s \mu_{is}^{\hat{l}}(t^r)}{r\sum_i w_i^{\hat{l}}} > \frac{C f_i(t_i|t^r)}{L\sum_i w_i^{\hat{l}}} \quad \text{(By (1),(9) and (33))} \\
&> \frac{C[\rho_i(t_i|\theta_k) - \eta]}{L\sum_i w_i^{\hat{l}}} \quad \text{(By (34))}.
\end{aligned}$$

which contradicts with (28).

Step 1.3: There exists an $\hat{r}_3 > 0$ such that for any $r > \hat{r}_3$, any $k \in J_m$, any $t^r \in B_k^r(\eta)$ and any $l \neq L+1$,

$$\pi^l(\mu(t^r)) \geq Q^l.$$

To prove this by contradiction, first suppose that for any $\hat{r}_3 > 0$ such that there exists an $r > \hat{r}_3$, a $k \in J_m$, a $\hat{l} \neq L+1$ and a $t^r \in B_k^r(\eta)$ such that

$$\pi^{\hat{l}}(\mu(t^r)) < Q^{\hat{l}}.$$

By (12) in assumption A.2, we have

$$\pi^{\hat{l}}(\mu(t^r)) < Q^{\hat{l}} \leq \frac{\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{\hat{l}}} P^r(\theta|\pi(\mu(t^r)), t_{is})}{\sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)],\theta)}{\partial x^{L+1}} P^r(\theta|\pi(\mu(t^r)), t_{is})}. \tag{35}$$

Note that the definition of $\pi^l(\mu(t^r))$ implies that there exists an agent $(i,s) \in N_r$ such that $\mu_{is}^{\hat{l}}(t^r) \leq \pi^{\hat{l}}(\mu(t^r))w_i^{\hat{l}}$. Choose $\hat{r}_3 > \hat{r}_2$. We conclude from step 1.2 that $x_{is}^{L+1}[\mu(t^r)] > 0$. For sufficiently small γ, consider $a_{is}(\gamma) \in A_i$ where

$$a_{is}(\gamma) = (\mu_{is}^1(t^r), \cdots, \mu_{is}^{\hat{l}}(t^r) + \gamma, \cdots, \mu_{is}^L(t^r)).$$

Similarly, let $x_{is}(r,\gamma) = x_{is}[\mu_{-is}(t^r), a_{is}(\gamma)]$ and $v_i(r,\gamma) = \sum_\theta u_i(x_{is}(r,\gamma),\theta)P^r(\theta|\pi(\mu(t^r)), t_{is})$. Note that $x_{is}(r,0) = x_{is}[\mu(t^r)]$ and $v_i(r,0) = \sum_\theta u_i(x_{is}[\mu(t^r)],\theta)P^r(\theta|\pi(\mu(t^r)), t_{is})$. We will show that γ can be chosen so that $a_{is}(\gamma)$ is an element of A_i and $v_i(r,\gamma) > v_i(r,0)$. This contradiction will complete the proof of step 1.3.
Next, define

$$g(\delta) = \frac{\delta}{1 - \frac{w_i^{\hat{l}}}{r\sum_i w_i^{\hat{l}}}}.$$

for each $\delta \geq 0$ and note that

$$x_{is}^l[\mu_{-is}(t^r), a_{is}(g(\delta))] = x_{is}^l[\mu(t^r)] \quad \text{for any } l \neq \hat{l};$$

$$x_{is}^{\hat{l}}[\mu_{-is}(t^r), a_{is}(g(\delta))] = \frac{\mu_{is}^{\hat{l}}(t^r) + g(\delta)}{\pi^{\hat{l}}(\mu(t^r)) + \frac{g(\delta)}{r\sum_i w_i^{\hat{l}}}};$$

$$x_{is}^{L+1}[\mu_{-is}(t^r), a_{is}(g(\delta))] = x_{is}^{L+1}[\mu(t^r)] - \delta.$$

Note $g(\delta) > 0$ for all $\delta > 0$ and $g(0) = 0$. Furthermore, g is differentiable with

$$g'(\delta) = \frac{1}{1 - \frac{w_i^{\hat{l}}}{r\sum_i w_i^{\hat{l}}}}.$$

Therefore, the derivative of $v_i(r, g(\delta))$ with respect to δ is

$$\frac{\partial v_i(r,g(\delta))}{\partial \delta} = \sum_\theta \left[\frac{\partial u_i(x_{is}(r,\delta),\theta)}{\partial x^{\hat{l}}} \frac{\partial x^{\hat{l}}}{\partial \delta} - \frac{\partial u_i(x_{is}(r,\delta),\theta)}{\partial x^{L+1}} \right] P^r(\theta|\pi(\mu(t^r)), t_{is}). \tag{36}$$

where

$$\frac{\partial x^{\hat{l}}}{\partial \delta} = \frac{\pi^{\hat{l}}(\mu(t^r)) - \frac{\mu_{is}^{\hat{l}}(t^r)}{r\sum_i w_i^{\hat{l}}}}{[\pi^{\hat{l}}(\mu(t^r)) + \frac{g(\delta)}{r\sum_i w_i^{\hat{l}}}]^2} g'(\delta).$$

Note that $\mu_{is}^{\hat{l}}(t^r) \leq \pi^l(\mu(t^r))w_i^{\hat{l}}$ implies

$$\frac{\partial x^{\hat{l}}}{\partial \delta}\Big|_{\delta=0} = \frac{1}{\pi^{\hat{l}}(\mu(t^r))} \frac{1 - \frac{\mu_{is}^{\hat{l}}(t^r)}{r\sum_i w_i^{\hat{l}}} \frac{1}{\pi^{\hat{l}}(\mu(t^r))}}{1 - \frac{w_i^{\hat{l}}}{r\sum_i w_i^{\hat{l}}}} \geq \frac{1}{\pi^{\hat{l}}(\mu(t^r))}.$$

Together with (35), at $\delta = 0$ we have

$$\begin{aligned} \sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)], \theta)}{\partial x^{L+1}} P^r(\theta|\pi(\mu(t^r)), t_{is}) &< \sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)], \theta)}{\partial x^{\hat{l}}} P^r(\theta|\pi(\mu(t^r)), t_{is}) \frac{1}{\pi^{\hat{l}}(\mu(t^r))} \\ &\leq \sum_\theta \frac{\partial u_i(x_{is}[\mu(t^r)], \theta)}{\partial x^{\hat{l}}} \frac{\partial x^{\hat{l}}}{\partial \delta}\Big|_{\delta=0} P^r(\theta|\pi(\mu(t^r)), t_{is}). \end{aligned}$$

So by (36) it follows that $\frac{\partial v_i(r,g(\delta))}{\partial \delta}|_{\delta=0} > 0$, which implies that there exists a $\delta^* > 0$ such that for any $0 < \delta < \delta^*$,

$$v_i(r, g(\delta)) > v_i(r, g(0)) = v_i(r, 0).$$

Therefore, choose $0 < \delta < \delta^*$ such that $x_{is}^{L+1}[\mu(t^r)] - \delta > 0$. It follows that $a_{is}(g(\delta)) \in A_i$ is feasible for agent (i, s) and $v_i(r, g(\delta)) > v_i(r, 0)$. This contradicts the assumption that μ is an ICSFM.

Step 1.4: Given the choice of $\hat{r}_1, \hat{r}_2$ and $\hat{r}_3$ in step 1.1-1.3, we let $\hat{r} = \max\{\hat{r}_1, \hat{r}_2, \hat{r}_3\}$. This completes the proof of step 1.

Step 2: For each r, each $(i, s) \in N_r$, each $a_i \in A_i$ and each $t^r \in T^r$,

$$\sum_\theta u_i(x_{is}[\mu(t^r)], \theta) P^r(\theta|q(t^r), t_{is}) \geq \sum_\theta u_i(x_{is}[\mu_{-is}(t^r), a_i)], \theta) P^r(\theta|q(t^r), t_{is}).$$

Since μ is an ICSFM, so for every function $\delta_i : \mathbb{R}^{L+1} \to A_i$, we have

$$\sum_{t_{-is}^r} \sum_\theta u_i(x_{is}[\mu(t^r)], \theta) P^r(\theta, t_{-is}^r|t_{is}) \geq \sum_{t_{-is}^r} \sum_\theta u_i(x_{is}[\mu_{-is}(t^r), \delta_i(\pi(\mu(t^r))], \theta) P^r(\theta, t_{-is}^r|t_{is}).$$

Now fix $t^r = (t_{-is}^r, t_{is}) \in T^r$ and $a_i \in A_i$ and define $\delta_i(\cdot) : \mathbb{R}_+^{L+1} \to A_i$ so that

$$\delta_i(\pi(\mu(\hat{t}_{-is}^r, t_{is}))) = \delta_i(q(\hat{t}_{-is}^r, t_{is}))) = \begin{cases} a_i & \text{if } q(\hat{t}_{-is}^r, t_{is}) = q(t_{-is}^r, t_{is}) \\ \mu_{is}(\hat{t}_{-is}^r, t_{is}) & \text{otherwise.} \end{cases}$$

Then it follows that

$$\sum_{\substack{\hat{t}_{-is} \\ :q(\hat{t}_{-is}^r, t_{is}) \\ = q(t_{-is}^r, t_{is})}} \sum_\theta u_i(x_{is}[\mu(\hat{t}_{-is}^r, t_{is})], \theta) P^r(\theta, \hat{t}_{-is}|t_{is}) \geq \sum_{\substack{\hat{t}_{-is} \\ :q(\hat{t}_{-is}^r, t_{is}) \\ = q(t_{-is}^r, t_{is})}} \sum_\theta u_i(x_{is}[\mu_{-is}(\hat{t}_{-is}^r, t_{is}), a_i)], \theta) P^r(\theta, \hat{t}_{-is}|t_{is}).$$

Note that

$$P^r(\theta|q(t_{-is}^r, t_{is}), t_{is}) = \frac{\sum_{\substack{\hat{t}_{-is} \\ :q(\hat{t}_{-is}^r, t_{is}) \\ = q(t_{-is}^r, t_{is})}} P^r(\theta, \hat{t}_{-is}|t_{is})}{\sum_{\substack{\hat{t}_{-is} \\ :q(\hat{t}_{-is}^r, t_{is}) \\ = q(t_{-is}^r, t_{is})}} P^r(\hat{t}_{-is}|t_{is})}.$$

If $q(\hat{t}^r_{-is}, t_{is}) = q(t^r_{-is}, t_{is})$, then $\pi(\mu(\hat{t}^r_{-is}, t_{is})) = \pi(\mu(t^r_{-is}, t_{is}))$. Since μ is adapted, $\mu_{is}(\hat{t}^r_{-is}, t_{is}) = \mu_{is}(t^r_{-is}, t_{is})$. Therefore $q(\hat{t}^r_{-is}, t_{is}) = q(t^r_{-is}, t_{is})$ implies that

$$\begin{aligned}
\pi^l(\mu(\hat{t}^r_{-is}, t_{is}), a_i) &= \frac{\sum_{i'}\sum_{s'} \mu^l_{i',s'}(\hat{t}^r_{-is}, t_{is}) + a^l_i - \mu_{is}(\hat{t}^r_{-is}, t_{is})}{r\sum_{i=1}^n w^l_i} \\
&= \frac{\sum_{i'}\sum_{s'} \mu^l_{i's'}(\hat{t}^r_{-is}, t_{is})}{r\sum_{i=1}^n w^l_i} + \frac{a^l_i - \mu_{is}(\hat{t}^r_{-is}, t_{is})}{r\sum_{i=1}^n w^l_i} \\
&= \pi^l(\mu(\hat{t}^r_{-is}, t_{is})) + \frac{a^l_i - \mu^l_{is}(\hat{t}^r_{-is}, t_{is})}{r\sum_{i=1}^n w^l_i} \\
&= \pi^l(\mu(t^r_{-is}, t_{is})) + \frac{a^l_i - \mu^l_{is}(t^r_{-is}, t_{is})}{r\sum_{i=1}^n w^l_i} \\
&= \frac{\sum_{i'}\sum_{s'} \mu^l_{i's'}(t^r_{-is}, t_{is}) + a^l_i - \mu_{is}(t^r_{-is}, t_{is})}{r\sum_{i=1}^n w^l_i} \\
&= \pi^l(\mu(t^r_{-is}, t_{is}), a_i).
\end{aligned}$$

Consequently, $q(\hat{t}^r_{-is}, t_{is}) = q(t^r_{-is}, t_{is})$ implies that

$$x^l_{is}[\mu_{-is}(\hat{t}^r_{-is}, t_{is}), a_i] = \frac{a^l_i}{\pi^l(\mu(\hat{t}^r_{-is}, t_{is}), a_i)} = \frac{a^l_i}{\pi^l(\mu(t^r_{-is}, t_{is}), a_i)} = x^l_{is}[\mu^r(t^r_{-is}, t_{is}), a_i] = x^l_{is}[\mu(t^r), a_i],$$

and (letting $a^l_i = \mu^l_{is}(t^r_{-is}, t_{is}) = \mu^l_{is}(\hat{t}^r_{-is}, t_{is})$), it follows that

$$x^l_{is}[\mu(\hat{t}^r_{-(is)}, t_{is})] = \frac{\mu^l_{is}(\hat{t}^r_{-is}, t_{is}))}{\pi^l(\mu(\hat{t}^r_{-is}, t_{is}))} = \frac{\mu^l_{is}(t^r_{-is}, t_{is})}{\pi^l(\mu^r(t^r_{-is}, t_{is}))} = x^l_{is}[\mu(t^r)].$$

Then we conclude that for any $a_i \in A_i$

$$\sum_\theta u_i(x_{is}[\mu(t^r)], \theta)P^r(\theta|q(t^r), t_{is}) \geq \sum_\theta u_i(x_{is}[\mu_{-is}(t^r), a_i)], \theta)P^r(\theta|q(t^r), t_{is}).$$

Step 3: First, for any $a_i \in A_i$ and $q \in \mathbb{R}^L_{++}$ define

$$y_{is}[a_i|q] \equiv (\frac{a^1_i}{q^1}, \cdots, \frac{a^L_i}{q^L}, w^{L+1}_i + \sum_{l=1}^L q^l w^l_i - \sum_{l=1}^L a^l_i).$$

In words, $y_{is}[a_i|q]$ is the allocation for agent (i, s) if he chooses action a_i and the market game price is exogenously given as q. Then we prove the following claim.
Claim 1: For any $\rho > 0$ and any $\eta \in (0, \eta^*)$, there exists an $\hat{r} > 0$ such that for any $r > \hat{r}$, any $k \in J_m$, any $t^r \in B^r_k(\eta)$, any $(i, s) \in N_r$, and any $a_i \in A_i$,

$$||x_{is}[\mu_{-is}(t^r), a_i] - y_{is}[a_i|\pi(\mu(t^r))]|| < \rho.$$

Here $||\cdot||$ is the $\mathbb{L}1$ norm.

Proof. Choose $\rho > 0$ and $\eta \in (0, \eta^*)$. By step 1, there exists an $\hat{r}_1 > 0$ such that for any $r > \hat{r}_1$, any $k \in J_m$, any $t^r \in B_k^r(\eta)$ and any $l \neq L+1$, we have $\pi^l(\mu(t^r)) \geq Q^l > 0$. Then note that

$$\pi^l(\mu_{-is}(t^r), a_i) = \pi^l(\mu(t^r)) + \frac{a_i^l - \mu_{is}^l(t^r)}{r \sum_i w_i^l}.$$

Since a_i^l and $\mu_{is}^l(t^r)$ are bounded, it follows that $\lim_{r\to\infty}[\pi^l(\mu_{-is}(t^r), a_i) - \pi^l(\mu(t^r))] = 0$. Therefore, there exists an $\hat{r}_2^{L+1} > 0$ such that for all $r > \hat{r}_2^{L+1}$,

$$|x_{is}^{L+1}[\mu_{-is}(t^r), a_i] - y_{is}^{L+1}[a_i|\pi(\mu(t^r))]| < \frac{\rho}{L+1}.$$

By Lemma 2, for each $l \neq L+1$, let $x_r = \pi^l(\mu_{-is}(t^r), a_i)$, $y_r = \pi^l(\mu(t^r))$ and $A = Q^l$, then it follows that there exists an $\hat{r}_2^l > 0$ such that for all $r > \hat{r}_2^l$,

$$|\frac{1}{\pi^l(\mu_{-is}(t^r), a_i)} - \frac{1}{\pi^l(\mu(t^r))}| < \frac{\rho}{w_i^{L+1}(L+1)}.$$

Since $a_i^l < w_i^{L+1}$, so it follows that

$$|x_{is}^l[\mu_{-is}(t^r), a_i] - \frac{a_i^l}{\pi^l(\mu(t^r))}| < \frac{\rho}{L+1}.$$

Therefore, there exists a $\hat{r} = \max\{\hat{r}_1, \hat{r}_2^1, \cdots, \hat{r}_2^{L+1}\} > 0$ such that each $r > \hat{r}$, each i, each $a_i \in A_i$, each $k \in J_m$ and each $t^r \in B_k^r(\eta)$,

$$||x_{is}[\mu_{-is}(t^r), a_i] - y_{is}[a_i|\pi(\mu^r(t^r))]|| < \sum_{l=1}^{L+1} \frac{\rho}{L+1} = \rho.$$

■

Step 4: For any $\eta \in (0, \eta^*)$, there exists an $\hat{r} > 0$ and a compact set $D \in \mathbb{R}^{L+1}$ such that for any $r > \hat{r}$, any $k \in J_m$, any $t^r \in B_k^r(\eta)$, any $(i, s) \in N_r$, and any $a_i \in A_i$,

$$x_{is}[\mu_{-is}(t^r), a_i] \in D;$$
$$y_{is}[a_i|\pi(\mu(t^r))] \in D.$$

To see this, choose $\eta \in (0, \eta^*)$. By step 1, there exists an $\hat{r}_1 > 0$ such that for any $r > \hat{r}_1$, any $k \in J_m$, any $t^r \in B_k^r(\eta)$ and any $l \neq L+1$, we have $\pi^l(\mu(t^r)) \geq Q^l > 0$. For simplicity, let $\bar{q} = \min_l Q^l$ and $\bar{w}^{L+1} = \max_i w_i^{L+1}$. Applying Claim 1 of step 3, there exists a $\hat{r}_2 > 0$ such that for all $r > \hat{r}_2$,

$$||x_{is}[\mu_{-is}(t^r), a_i] - y_{is}[a_i|\pi(\mu(t^r))]|| < \frac{\bar{w}^{L+1}}{\bar{q}}. \tag{37}$$

Since $\pi^l(\mu(t^r)) \geq \bar{q}$ and $a_i^l \leq \bar{w}^{L+1}$, it follows that for all l,

$$\frac{a_i^l}{\pi^l(\mu(t^r))} \leq \frac{\bar{w}^{L+1}}{\bar{q}}. \tag{38}$$

For good $L+1$, we have

$$w_i^{L+1} + \sum_{l=1}^{L} \pi^l(\mu(t^r))w_i^l - \sum_{l=1}^{L} a_i^l \leq w_i^{L+1} + \sum_{l=1}^{L} K^l w_i^l := M_i. \tag{39}$$

Let $\hat{r} = \max\{\hat{r}_1, \hat{r}_2\}$. Then for all $r > \hat{r}$, let $M \equiv \max_i M_i$ and define the compact set

$$D = \{x \in \mathbb{R}_+^{L+1} | x^l \leq \frac{2\bar{w}^{L+1}}{\bar{q}} \text{ for all } l \text{ and } x^{L+1} \leq M + \frac{\bar{w}^{L+1}}{\bar{q}}\}.$$

Note by (38) and (39), we have $y_{is}[a_i|\pi(\mu(t^r))] \in D$. Moreover, from (37), $x_{is}^l[\mu_{-is}(t^r), a_i] \leq \frac{2\bar{w}^{L+1}}{\bar{q}}$ and $x_{is}^{L+1}[\mu_{-is}(t^r), a_i] \leq M + \frac{\bar{w}^{L+1}}{\bar{q}}$. Therefore, $x_{is}[\mu_{-is}(t^r), a_i]$ is also in D.

Step 5: For any $\rho > 0$ and any $\eta \in (0, \eta^*)$, there exists an $\hat{r} > 0$ such that for any $r > \hat{r}$, any $k \in J_m$, any $t^r \in B_k^r(\eta)$, any $(i,s) \in N_r$, and any $a_i \in A_i$,

$$|u_i(x_{is}[\mu_{-is}(t^r), a_i], \theta) - u_i(y_{is}[a_i|\pi(\mu(t^r))], \theta)| < \rho. \tag{40}$$

Proof. Choose $\rho > 0$ and $\eta \in (0, \eta^*)$. By step 4, there exists an $\hat{r}_1 > 0$ and a compact set $D \in \mathbb{R}^{L+1}$ such that for any $r > \hat{r}_1$, any $k \in J_m$, any $t^r \in B_k^r(\eta)$, any $(i,s) \in N_r$ and any $a_i \in A_i$,

$$x_{is}[\mu_{-is}(t^r), a_i] \in D;$$
$$y_{is}[a_i|\pi(\mu(t^r))] \in D.$$

By the uniform continuity of $u_i(\cdot, \theta)$ in D for each θ, there exists a δ such that for any $x_i \in D$ and any $y_i \in D$,

$$||x_i - y_i|| < \delta \Rightarrow ||u_i(x_i, \theta) - u_i(y_i, \theta)|| < \rho. \tag{41}$$

By claim 1 of step 3, there exists a $\hat{r}_2$ such that for any $r > \hat{r}_2$,

$$||x_{is}[\mu_{-is}(t^r), a_i] - y_{is}[a_i|\pi(\mu^r(t^r))]|| < \delta. \tag{42}$$

Hence, let $\hat{r} > \max\{\hat{r}_1, \hat{r}_2\}$. Then, for any $r > \hat{r}$, by (41) and (42) we have

$$|u_i(x_{is}[\mu_{-is}(t^r), a_i], \theta) - u_i(y_{is}[a_i|\pi(\mu(t^r))], \theta)| < \rho.$$

■

Step 6: Choose $\varepsilon > 0$. We now show that $(\pi(\mu(\cdot)), \{x_{is}[\mu(\cdot)]\}_{(i,s)\in N_r})$ is a type symmetric $\varepsilon-$REE. Pick $\eta < \min\{\eta^*, \varepsilon\}$. By Lemma 1.i), there exists a $\hat{r}_1$ such that for any $r > \hat{r}_1$ and associated partition $\Pi^r(\eta)$,

$$Prob\{\tilde{t}^r \in \cup_{k=1}^m B_k^r(\eta)\} \geq 1 - \eta \geq 1 - \varepsilon. \tag{43}$$

Moreover, step 1.2 implies that there exists an $\hat{r}_2$ such that for any $r > \hat{r}_2$, $k \in J_m$, $t^r \in B_k^r(\eta)$, $(i,s) \in N_r$, and $a_i \in A_i$,

$$\sum_{l=1}^{L} \mu_{is}^l(t^r) \leq C < w_i^{L+1}.$$

Choose a constant $\alpha \in (0,1)$ that is close to 0 such that

$$\alpha[\sum_{l=1}^{L} K^l w_i^l + w_i^{L+1}] + (1-\alpha)C < w_i^{L+1}. \tag{44}$$

Applying step 5, there exists an $\hat{r}_3 > 0$ such that for any $r > \hat{r}_3$, $k \in J_m$, $t^r \in B_k^r(\eta)$, $(i,s) \in N_r$, and $a_i \in A_i$,

$$|u_i(x_{is}[\mu_{-is}(t^r), a_i], \theta) - u_i(y_{is}[a_i|\pi(\mu(t^r))], \theta)| < \alpha\varepsilon. \tag{45}$$

Therefore, let $\hat{r} = \max\{\hat{r}_1, \hat{r}_2, \hat{r}_3\}$. Fix $r > \hat{r}$. Let $S^r = \cup_{k \in J_m} B_k^r(\eta)$. By (43), it follows that (7) holds. Then to show $(\pi(\mu(\cdot)), \{x_{is}[\mu(\cdot)]\}_{(i,s)\in N_r})$ satisfies (2),(4) and (5), choose $t^r, \hat{t}^r \in S^r$, $(i,s) \in N_r$ and $(i,s') \in N_r$. If

$f(t^r) = f(\hat{t}^r)$, then by type symmetry we have

$$\pi^l(\mu(t^r)) = \frac{\sum_i \sum_{\tau_i} \sum_{s:t_{is}=\tau_i} \mu^l_{is}(t^r_{-is}, \tau_i)}{r \sum_j w^l_j} = \frac{\sum_i \sum_{\tau_i} \mu^l_{is}(t^r_{-is}, \tau_i) f_i(\tau_i|t^r)}{r \sum_j w^l_j} = \frac{\sum_i \sum_{\tau_i} \mu^l_{is}(\hat{t}^r_{-is}, \tau_i) f_i(\tau_i|t^r)}{r \sum_j w^l_j} = \pi^l(\mu(\hat{t}^r))$$

If $q(t^r) = q(\hat{t}^r), t_{is} = \hat{t}_{is}$, then $\pi(\mu(t^r)) = \pi(\mu(\hat{t}^r)), t_{is} = \hat{t}_{is}$ implying that $\mu_{is}(\hat{t}^r) = \mu_{is}(t^r)$ since μ is adapted. Therefore,

$$z^l_{is}(t^r) = \frac{\mu^l_{is}(t^r)}{q^l(t^r)} = \frac{\mu^l_{is}(\hat{t}^r)}{q^l(\hat{t}^r)} = z^l_{is}(\hat{t}^r).$$

Moreover,

$$\sum_i \sum_s z^l_{is}(t^r) = \sum_i \sum_s \frac{\mu^l_{is}(t^r)}{\pi^l(\mu(t^r))} = \frac{\sum_i \sum_s \mu^l_{is}(t^r)}{\sum_i \sum_s \mu^l_{is}(t^r)} \left[r \sum_i w^l_i \right] = r \sum_i w^l_i.$$

Next, choosing $t^r \in S^r$, $(i, s) \in N_r$ and $\xi_i \in \beta_i(q(t^r))$, we are left to show that

$$\sum_\theta u_i(z_{is}(t^r), \theta) P^r(\theta|q(t^r), t_{is}) \geq \sum_\theta u_i(\xi_i, \theta) P^r(\theta|q(t^r), t_{is}) - \varepsilon.$$

Since $\xi_i \in \beta_i(q(t^r))$, applying step 1 again we have

$$\sum_{l=1}^L q^l(t^r)\xi^l_i \leq \sum_{l=1}^L q^l(t^r) w^l_i + w^{L+1}_i \leq \sum_{l=1}^L K^l w^l_i + w^{L+1}_i.$$

By (44), it follows that

$$\sum_{l=1}^L [\alpha q^l(t^r)\xi^l_i + (1-\alpha)\mu^l_{is}(t^r)] < w^{L+1}_i.$$

Defining

$$\eta_{is}(t^r, \alpha) = \alpha q(t^r)\xi_i + (1-\alpha)\mu_{is}(t^r),$$

we have

$$\sum_{l=1}^l \eta^l_{is}(t^r, \alpha) < w^{L+1}_i$$

implying $\eta_{is}(t^r, \alpha) \in A_i$. Also, note that

$$y_{is}[\eta_{is}(t^r, \alpha)|\pi(\mu(t^r))] = \alpha\xi_i + (1-\alpha)x_{is}[\mu(t^r)]. \tag{46}$$

Then it follows that

$$\begin{aligned}
&\sum_\theta u_i(x_{is}[\mu(t^r)], \theta) P^r(\theta|q(t^r), t_{is}) \\
&\geq \sum_\theta u_i(x_{is}[\mu_{-is}(t^r), \eta_{is}(t^r, \alpha)], \theta) P^r(\theta|q(t^r), t_{is}) \quad \text{(Applying Step 2)} \\
&> \sum_\theta u_i(y_{is}[\eta_{is}(t^r, \alpha)|\pi(\mu(t^r))], \theta) P^r(\theta|q(t^r), t_{is}) - \alpha\varepsilon \quad \text{(By (45))} \\
&= \sum_\theta u_i(\alpha\xi_i + (1-\alpha)x_{is}[\mu(t^r)], \theta) P^r(\theta|q(t^r), t_{is}) - \alpha\varepsilon \text{ (By (46))} \\
&\geq \alpha \sum_\theta u_i(\xi_i, \theta) P^r(\theta|q(t^r), t_{is}) + (1-\alpha) \sum_\theta u_i(x_{is}[\mu(t^r)], \theta) P^r(\theta|q(t^r), t_{is}) - \alpha\varepsilon \quad \text{(Concavity).}
\end{aligned}$$

We conclude that

$$\sum_{\theta} u_i(x_{is}[\mu(t^r)], \theta) P^r(\theta | q(t^r), t_{is}) \geq \sum_{\theta} u_i(\xi_i, \theta) P^r(\theta | q(t^r), t_{is}) - \varepsilon.$$

Appendix C: Proof of Theorem 2

Step 1: For each l, each r, and each $t^r \in T^r$,

$$Q^l \leq q^l(t^r) \leq K^l.$$

To see this, first we show that $q^l(t^r) \leq K^l$. Suppose that there exist a l, an r, and a $t^r \in T^r$ such that

$$q^l(t^r) > K^l.$$

Together with (12) in assumption A.2, we have

$$q^l(t^r) > K^l \geq \frac{\sum_{\theta} \frac{\partial u_i(x_{is},\theta)}{\partial x^l} P^r(\theta | q(t^r), t_{is})}{\sum_{\theta} \frac{\partial u_i(x_{is},\theta)}{\partial x^{L+1}} P^r(\theta | q(t^r), t_{is})}. \tag{47}$$

By the market clearing condition for good l at t^r, there exists an agent $(i, s) \in N_r$ such that $z_{is}^l(t^r) > 0$. Consider the following feasible allocation $z_{is}(t^r, \gamma) \in \beta_i(q(t^r))$ for agent (i, s):

$$z_{is}(t^r, \gamma) = (z_{is}^1(t^r), \cdots, z_{is}^l(t^r) - \frac{\gamma}{q^l(t^r)}, \cdots, z_{is}^L(t^r), z_{is}^{L+1}(t^r) + \gamma).$$

where $0 < \gamma < z_{is}^l(t^r) q^l(t^r)$. Let $v_i(r, \gamma) = \sum_{\theta \in \Theta} u_i(z_{is}(t^r, \gamma)) P^r(\theta | q(t^r), t_{is})$. Then by (47) we have

$$\frac{\partial v_i(r,\gamma)}{\partial \gamma}\Big|_{\gamma=0} = \sum_{\theta} \left[-\frac{\partial u_i(x_{is},\theta)}{\partial x^l} \right] P^r(\theta | q(t^r), t_{is}) + \sum_{\theta} \left[\frac{\partial u_i(x_{is},\theta)}{\partial x^{L+1}} q^l(t^r) \right] P^r(\theta | q(t^r), t_{is}) > 0.$$

This contradicts with the assumption that $z_{is}(t^r)$ is a part of type symmetric REE. Similarly, now we show that $q^l(t^r) \geq Q^l$. Suppose that there exist a l, an r, and a $t^r \in S^r$ such that

$$q^l(t^r) < Q^l.$$

Together with (12) in assumption A.2, we have

$$q^l(t^r) < Q^l \leq \frac{\sum_{\theta} \frac{\partial u_i(x_{is},\theta)}{\partial x^l} P^r(\theta | q(t^r), t_{is})}{\sum_{\theta} \frac{\partial u_i(x_{is},\theta)}{\partial x^{L+1}} P^r(\theta | q(t^r), t_{is})}. \tag{48}$$

By the market clearing condition for good l at t^r, there exists an agent $(i, s) \in N_r$ such that $z_{is}^{L+1}(t^r) > 0$. Consider the following feasible allocation for agent (i, s):

$$z_{is}(t^r, \gamma) = (z_{is}^1(t^r), \cdots, z_{is}^l(t^r) + \frac{\gamma}{q^l(t^r)}, \cdots, z_{is}^L(t^r), z_{is}^{L+1}(t^r) - \gamma).$$

where $0 < \gamma < z_{is}^{L+1}(t^r)$. Let $v_i(r, \gamma) = \sum_{\theta \in \Theta} u_i(z_{is}(t^r, \gamma)) P^r(\theta | q(t^r), t_{is})$. Then by (48) we have

$$\frac{\partial v_i(r,\gamma)}{\partial \gamma}\Big|_{\gamma=0} = \sum_{\theta} \left[\frac{\partial u_i(x_{is},\theta)}{\partial x^l} \right] P^r(\theta | q(t^r), t_{is}) + \sum_{\theta} \left[-\frac{\partial u_i(x_{is},\theta)}{\partial x^{L+1}} q^l(t^r) \right] P^r(\theta | q(t^r), t_{is}) > 0.$$

This contradicts with the assumption that $z_{is}(t^r)$ is a part of an REE.

Step 2: For any $\rho > 0$, there exists an $\hat{r} > 0$ and a $\gamma \in (0, \rho)$ such that for any $r > \hat{r}$, any $(i, s) \in N_r$, any $t_{is} \in T_i$ and $k \in J_m$ the following are true:
i)
$$\sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_0(\gamma)}} P^r(t^r_{-is}|t_{is}) \leq \rho. \tag{49}$$

ii) For any $t'_i \in T_i$,
$$\sum_{k=1}^{m} \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\gamma) \\ (t^r_{-is}, t'_i) \notin B^r_k(\gamma) \cup B^r_0(\gamma)}} P^r(t^r_{-is}|t_{is}) \leq \rho. \tag{50}$$

iii) For any $t^r \in B^r_k(\gamma)$,
$$(1 - P^r(\theta_k|t^r)) + \sum_{\kappa \neq k} P^r(\theta_\kappa|t^r) \leq \rho. \tag{51}$$

iv) For any $M^r \in T^r$,
$$\sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\gamma) \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(t^r_{-is}|t_{is}) \leq \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(\theta_k, t^r_{-is}|t_{is}) + \rho \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\gamma) \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(t^r_{-is}|t_{is}). \tag{52}$$

v) For any $t'_i \in T_i$,
$$\sum_{k \in J_m} \sum_{\substack{t^r_{-is} \\ :(t^r_{-is}, t_{is}) \in B^r_k(\gamma) \\ (t^r_{-is}, t'_i) \in B^r_0(\gamma)}} P^r(t^r_{-is}|t_{is}) \leq \rho. \tag{53}$$

Proof. Choose $\rho > 0$. By Lemma 1.ii) and iii), there exists an r'_1 such that for any $r > r'_1$ and associated partition $\Pi^r(\rho)$, (49) - (51) hold. Next, applying Lemma 1.i) again, there exists a r'_2 such that for any $r > r'_2$ and associated partition $\Pi^r(\frac{\rho}{m})$, $t^r \in B^r_k(\frac{\rho}{m})$,
$$P^r(\theta_j|t^r) \leq \frac{\rho}{m} \quad \text{for all } j \neq k. \tag{54}$$

Choose $M^r \in T^r$ and it follows that

$$\begin{aligned}
\sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\frac{\rho}{m}) \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(t^r_{-is}|t_{is}) &= \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\frac{\rho}{m}) \\ (t^r_{-is}, t_{is}) \in M^r}} [P^r(\theta_k, t^r_{-is}|t_{is}) + \sum_{j \neq k} P^r(\theta_j, t^r_{-is}|t_{is})] \\
&\leq \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(\theta_k, t^r_{-is}|t_{is}) + \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\frac{\rho}{m}) \\ (t^r_{-is}, t_{is}) \in M^r}} [\sum_{j \neq k} P^r(\theta_j|t^r_{-is}, t_{is})] P^r(t^r_{-is}|t_{is}) \\
&\leq \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\frac{\rho}{m}) \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(\theta_k, t^r_{-is}|t_{is}) + \rho \frac{m-1}{m} \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\frac{\rho}{m}) \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(t^r_{-is}|t_{is}) \quad \text{(By (54))} \\
&\leq \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\frac{\rho}{m}) \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(\theta_k, t^r_{-is}|t_{is}) + \rho \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_{is}) \in B^r_k(\frac{\rho}{m}) \\ (t^r_{-is}, t_{is}) \in M^r}} P^r(t^r_{-is}|t_{is})
\end{aligned}$$

Next, by the analysis in the proof of Theorem 2 in McLean and Postlewaite (2002)(see Page 2449), there exists $\lambda > 0$ and an integer r_3' such that for any $r > r_3'$,

$$||f(t^r) - P^r(\cdot|\theta_k)|| < \lambda \Rightarrow ||P^r(\cdot|t^r) - I_{\theta_k}|| < \frac{\rho}{2} \quad \text{for all } t^r \text{ and } k \in J_m; \tag{55}$$

$$||f(t^r_{-is}, t_{is}) - f(t^r_{-is}, t_i')|| < \frac{\lambda}{2} \qquad \text{for all } t_i, t_i' \in T_i \text{ and all } t^r \text{ and all } i; \tag{56}$$

and

$$Prob\{t^r_{-is} : ||f(t^r_{-is}, \tilde{t}_{is}) - P^r(\cdot|\theta_k)|| < \frac{\lambda}{2}|\tilde{t}_{is} = t_{is}, \tilde{\theta} = \theta_k\} > 1 - \rho \quad \text{for all } t_i, t_i' \in T_i \text{ and } k \in J_m. \tag{57}$$

Choose $t_i, t_i' \in T_i$, and $r > \hat{r}$. Note from (55), it follows that for all t^r and $k \in J_m$,

$$||P^r(\cdot|t^r) - I_{\theta_k}|| \geq \frac{\rho}{2} \Rightarrow ||f(t^r) - P^r(\cdot|\theta_k)|| \geq \lambda. \tag{58}$$

Combining (56) and (58), we have

$$\begin{aligned} ||f(t^r_{-is}, t_{is}) - P^r(\cdot|\theta_k)|| &= ||f(t^r_{-is}, t_{is}) - f(t^r_{-is}, t_i') + f(t^r_{-is}, t_i') - P^r(\cdot|\theta_k)|| \\ &\geq |\{||f(t^r_{-is}, t_i') - P^r(\cdot|\theta_k)||\} - \{||f(t^r_{-is}, t_{is}) - f(t^r_{-is}, t_i')||\}| \\ &\geq \frac{\lambda}{2}. \end{aligned} \tag{59}$$

Then, it follows that

$$\begin{aligned} & Prob\{t^r_{-is} : ||P^r(\cdot|t^r_{-is}, t_i') - I_{\theta_k}|| > \frac{\rho}{2} \text{ for all } k|\tilde{t}^r_{is} = t_{is}, \tilde{\theta} = \theta_k\} \\ &\quad \leq Prob\{t^r_{-is} : ||P^r(\cdot|t^r_{-is}, t_i') - I_{\theta_k}|| > \frac{\rho}{2}|\tilde{t}^r_{is} = t_{is}, \tilde{\theta} = \theta_k\} \\ &\quad = Prob\{t^r_{-is} : ||P^r(\cdot|t^r_{-is}, t_i') - I_{\theta_k}|| > \frac{\rho}{2} \text{ and } ||f(t^r_{-is}, t_{is}) - f(t^r_{-is}, t_i')|| < \frac{\lambda}{2}|\tilde{t}^r_{is} = t_{is}, \tilde{\theta} = \theta_k\} \\ &\quad \leq Prob\{t^r_{-is} : ||f(t^r) - P^r(\cdot|\theta_k)|| \geq \lambda \text{ and } ||f(t^r_{-is}, t_{is}) - f(t^r_{-is}, t_i')|| < \frac{\lambda}{2}|\tilde{t}^r_{is} = t_{is}, \tilde{\theta} = \theta_k\} \quad \text{(By (58))} \\ &\quad \leq Prob\{t^r_{-is} : ||f(t^r_{-is}, \tilde{t}^r_{is}) - P^r(\cdot|\theta_k)|| \geq \frac{\lambda}{2}|\tilde{t}^r_{is} = t_{is}, \tilde{\theta} = \theta_k\} \quad \text{(By (59))} \\ &\quad \leq \rho \quad \text{(By (57))}. \end{aligned}$$

Therefore,

$$Prob\{t^r_{-is} : ||P^r(\cdot|t^r_{-is}, t_i') - I_{\theta_k}|| > \frac{\rho}{2} \text{ for all } k|\tilde{t}^r_{is} = t_{is}\} \leq \rho.$$

Note that

$$\sum_{\substack{t^r_{-is} \\ :(t^r_{-is}, t_i') \in B^r_0(\frac{\rho}{2})}} P^r(t^r_{-is}|t_{is}) = Prob\{t^r_{-is} : ||P^r(\cdot|t^r_{-is}, t_i') - I_{\theta_k}|| > \frac{\rho}{2} \text{ for all } k|\tilde{t}^r_{is} = t_{is}\}.$$

So it follows that

$$\sum_{k \in J_m} \sum_{\substack{t^r_{-is} \\ :(t^r_{-is}, t_{is}) \in B^r_k(\frac{\rho}{2}) \\ (t^r_{-is}, t_i') \in B^r_0(\frac{\rho}{2})}} P^r(t^r_{-is}|t_{is}) = \sum_{\substack{t^r_{-is} \\ :(t^r_{-is}, t_{is}) \notin B^r_0(\frac{\rho}{2}) \\ (t^r_{-is}, t_i') \in B^r_0(\frac{\rho}{2})}} P^r(t^r_{-is}|t_{is}) \leq \sum_{\substack{t^r_{-is} \\ :(t^r_{-is}, t_i') \in B^r_0(\frac{\rho}{2})}} P^r(t^r_{-is}|t_{is}) \leq \rho.$$

Therefore, choose $\hat{r} = \max\{r_1', r_2', r_3'\}$ and $\gamma = \min\{\frac{\rho}{m-1}, \frac{\rho}{2}\}$. This complete the proof of step 2. ■

Step 3: For any $\rho > 0$, there exists an $\hat{r} > 0$ and a $\gamma > 0$ such that for any $r > \hat{r}$, any $k \in J_m$,

$$t^r, \hat{t}^r \in B_k^r(\gamma) \Rightarrow ||q(t^r) - q(\hat{t}^r)|| < \rho.$$

To see this, by assumption A.3, there exists a r_1' and $\delta > 0$ such that for any $r > r_1'$,

$$||f(t^r) - f(\hat{t}^r)|| < \delta \Rightarrow ||q(t^r) - q(\hat{t}^r)|| < \rho. \tag{60}$$

Next, by lemma 1.iv) and triangle inequality, there exists an r_2' and $\frac{\delta}{2}$ such that for any $r > r_2'$ and associated partition $\Pi^r(\frac{\delta}{2})$, $k \in J_m$,

$$t^r, \hat{t}^r \in B_k^r(\frac{\delta}{2}) \Rightarrow ||f(t^r) - f(\hat{t}^r)|| < \delta. \tag{61}$$

Therefore, by (60) and (61) there exists an $\hat{r} = \max\{r_1', r_2'\}$ and $\gamma = \frac{\delta}{2}$ such that for any $r > \hat{r}$ and associated partition $\Pi^r(\gamma)$, $k \in J_m$,

$$t^r, \hat{t}^r \in B_k^r(\gamma) \Rightarrow ||q(t^r) - q(\hat{t}^r)|| < \rho.$$

Step 4: In this step, we construct the mechanism $\mu : T^r \to A^r$. First, choose a $\bar{t}^r \in B_k^r(\gamma)$ for each $k \in J_m$, denoted $\bar{t}^{r,k}$. Let $\bar{q}_k^r = q(\bar{t}^{r,k})$ be the corresponding type symmetric REE price at $\bar{t}^{r,k}$. Then the mechanism μ is constructed as follows: for each (i, s) and each $l \neq L+1$,

$$\mu_{i,s}^l(t^r) = \begin{cases} \bar{q}_k^{r,l} z_{i,s}^l(t^r) & \text{if } t^r \in B_k^r(\gamma) \\ q^l(t^r) z_{is}^l(t^r) & \text{if } t^r \in B_0^r(\gamma). \end{cases}$$

Note that the price determined by the mechanism is the following: for $t^r \in B_k^r$,

$$\pi^l(\mu(t^r)) = \frac{\sum_{i=1}^{n}\sum_{s=1}^{r} \mu_{is}^l(t^r)}{r\sum_{i=1}^{n} w_i^l} = \frac{\bar{q}_k^r \sum_{s=1}^{r}\sum_{i=1}^{n} z_{is}^l(t^r)}{r\sum_{i=1}^{n} w_i^l} = \bar{q}_k^r.$$

and the corresponding allocation is

$$\begin{aligned} x_{is}^l[\mu(t^r)] &= \frac{\mu_{is}^l(t^r)}{\pi^l(\mu^r(t^r))} = \frac{\bar{q}_k^r z_{is}^l(t^r)}{\bar{q}_k^r} = z_{is}^l(t^r) \qquad \text{if } l \neq L+1; \\ x_{is}^{L+1}[\mu(t^r)] &= w_i^{L+1} + \sum_{l=1}^{L} \pi^l(\mu(t^r)) w_i^l - \sum_{l=1}^{L} \mu_{is}^l(t^r) \\ &= w_i^{L+1} + \sum_{l=1}^{L} \bar{q}_k^r w_i^l - \sum_{l=1}^{L} \bar{q}_k^r z_{is}^l(t^r). \end{aligned}$$

To proceed, we first prove the following claim.
Claim: There exists an $\hat{r} > 0$ and a compact set D such that for any $r > \hat{r}$, any $(i, s) \in N_r$, any $a_i \in A_i$ and any $t^r \in T^r$,

$$\begin{aligned} x_{is}[\mu_{-is}(t^r), a_i] &\in D; \\ y_{is}[a_i | q(t^r)] &\in D. \end{aligned}$$

Proof. Choose $(i, s) \in N_r$ and $t^r \in T^r$. First, step 1 implies that $\pi(\mu(t^r)) \in [Q, K]$. Let $\bar{q} = \min_l Q^l$. Then it follows that for any $y_i \in \beta_i(q(t^r))$ we have

$$\sum_{l=1}^{L} q^l(t^r) y_i^l + y_i^{L+1} \leq \sum_{l=1}^{L} q^l(t^r) w_i^l + w_i^{L+1} \Rightarrow y_i^l \leq \sum_{l=1}^{L} \frac{K^l}{\bar{q}} w_i^l + \frac{w_i^{L+1}}{\bar{q}} \text{ for all } l \neq L+1 \text{ and } y_i^{L+1} \leq \sum_{l=1}^{L} K^l w_i^l + w_i^{L+1}.$$

Define $M_i \equiv \max\{\sum_{l=1}^{L} \frac{K^l}{\bar{q}} w_i^l + \frac{w_i^{L+1}}{\bar{q}}, \sum_{l=1}^{L} K^l w_i^l + w_i^{L+1}\}$ and $M \equiv \max_i\{M_i\}$. Therefore it follows that $y_{is}^l[a_i|q(t^r)] \leq M$ for all l. Next, following the same analysis in the proof of Claim 1 in the proof of Theorem 1, there exists a $\hat{r}$ such that for any $r > \hat{r}$,

$$||x_{is}[\mu_{-is}(t^r), a_i] - y_{is}[a_i|q(t^r)]|| < M. \tag{62}$$

Defining a compact set D as follow:

$$D = \{x \in \mathbb{R}_+^{L+1} | x^l \leq 2M\}.$$

It follows that $x_{is}[\mu_{-is}(t^r), a_i] \in D$ and $y_{is}[a_i|q(t^r)] \in D$. ■

Step 5: For any $\rho > 0$, there exists an $\hat{r} > 0$ and a $\gamma > 0$ such that for any $r > \hat{r}$, $k \in J_m$, $\theta_k \in \Theta$, and $a_i \in A_i$ the following are true:

$$u_i(x_{i,s}[\mu(t^r)], \theta_k) - u_i(z_{i,s}(t^r), \theta_k) \geq -\rho; \tag{63}$$

$$u_i(x_{is}[\mu_{-is}(t^r), a_i]), \theta_k) - u_i(y_{is}[a_i|q(t^r)], \theta_k) \geq -\rho; \tag{64}$$

$$u_i(x_{is}[\mu_{-is}(t^r), a_i]), \theta_k) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t'_i), a_i]), \theta_k) \geq -\rho. \tag{65}$$

whenever $t^r = (t^r_{-is}, t_{is}) \in B_k^r(\gamma)$ and $(t^r_{-is}, t'_i) \in B_k^r(\gamma)$.

Proof. Choose $\rho > 0$. Applying the Claim in step 4, we have that there exists an r'_1 and a compact set D such that for any $r > r'_1$, $a_i \in A_i$ and $t^r \in T^r$,

$$x_{is}[\mu_{-is}(t^r), a_i] \in D;$$
$$y_{is}[a_i|q(t^r)] \in D.$$

Also, $z_{i,s}(t^r) \in D$. Since $u_i(\cdot, \theta)$ is uniformly continuous in D for each θ, It follows that there exists a $\delta > 0$ such that for any $x_i \in D$, $y_i \in D$ and $\theta \in \Theta$,

$$||x_i - y_i|| < \delta \Rightarrow |u_i(x_i, \theta) - u_i(y_i, \theta)| < \rho. \tag{66}$$

By step 3 and Lemma 2, there exists an $r'_2 > 0$ and a $\delta > 0$ such that for any $r > r'_2$, $t^r = (t^r_{-is}, t_i) \in B_k^r(\delta)$, $k \in J_m$ and $a_i \in A_i$,

$$||x_{i,s}[\mu(t^r)] - z_{i,s}(t^r)|| < \frac{\delta}{2}; \tag{67}$$

$$||y_{is}[a_i|q(t^r_{-is}, t_i)] - y_{is}[a_i|\bar{q}_k^r]|| < \frac{\delta}{2}. \tag{68}$$

To show (64). Note that by the Claim 1 of the proof of Theorem 1 there exists an $r'_3 > 0$ such that for any $r > r'_3$ and $t^r \in B_k^r(\delta)$,

$$||x_{is}[\mu_{-is}(t^r), a_i] - y_{is}[a_i|\bar{q}_k^r]|| < \frac{\delta}{2}. \tag{69}$$

It follows from (66) that for any $r > r'_3$ and $t^r \in B_k^r(\delta)$,

$$u_i(y_{is}[a_i|q(t^r)], \theta) - u_i(x_{is}[\mu_{-is}(t^r), a_i]), \theta) \geq -\rho.$$

Consequently, (63) also hold. To show (65). Note that for any $(t^r_{-is}, t_i) \in B_k^r(\delta)$ and any $(t^r_{-is}, t'_i) \in B_k^r(\delta)$, we have

$$y_{is}[a_i|\pi(\mu(t^r_{-is}, t_i))] = y_{is}[a_i|\bar{q}_k^r] = y_{is}[a_i|\pi(\mu(t^r_{-is}, t'_i))]$$

Then by (69) and triangle inequality we have

$$||x_{is}[\mu_{-is}(t^r), a_i] - x_{is}[\mu_{-is}(t^r_{-is}, t'_i), a_i]|| < \delta.$$

Then by (66), it follows that for any $r > r'_4$,

$$u_i(x_{is}[\mu_{-is}(t^r), a_i]), \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t'_i), a_i]), \theta) \geq -\rho.$$

Therefore, there exists an $\hat{r} = \max\{r_1', r_2', r_3'\}$ and a $\gamma = \delta$ such that for any $r > \hat{r}$, θ and $a_i \in A_i$, (64) and (65) hold. ■

Step 6: There exists an $\hat{r} > 0$ and a $\gamma > 0$ such that for any $r > \hat{r}$, any $(i,s) \in N_r$, any $k \in J_m$ and any $t^r \in B_k^r(\gamma)$,

$$\sum_{l=1}^{L} \mu_{is}^l(t^r) < w_i^{L+1}.$$

To see this, first by (13) in assumption A.2 we have that for each $\theta \in \Theta$, each $i \in N$ and each $t_i \in T_i$,

$$w_i^{L+1} > \frac{L \max_{l \neq L+1}[K^l \sum_{i \in N} w_i^l]}{\rho_i(t_i|\theta)}.$$

By continuity, choose $\gamma > 0$ so that for each $\theta \in \Theta$, each $i \in N$ and each $t_i \in T_i$,

$$w_i^{L+1} > \frac{L \max_{l \neq L+1}[K^l \sum_{i \in N} w_i^l]}{\rho_i(t_i|\theta) - \gamma}. \tag{70}$$

Then Lemma 1.i) and iv) imply that there exists an $\hat{r}$ such that for any $r > \hat{r}$ and associated partition $\Pi^r(\gamma)$, $i \in N$, $k \in J_m$, $t^r \in B_k^r(\gamma)$ and $t_i \in T_i$,

$$Prob\{\tilde{t}^r \in \cup_{k \in J_m} B_k^r(\gamma)\} \geq 1 - \gamma, \tag{71}$$

and

$$f_i(t_i|t^r) > \rho_i(t_i|\theta_k) - \gamma. \tag{72}$$

Choose $t^r \in B_k^r$ and $(i,s) \in N_r$. Suppose $t_{is} = t_i$. By the market clearing condition for t^r and the type symmetry condition (6), it follows that for each l,

$$r \sum_i w_i^l = \sum_i \sum_s z_{is}^l(t^r) = \sum_i \sum_{\tau_i \in T_i} \sum_{s: t_{is} = \tau_i} z_{is}^l(t_{-is}^r, \tau_i) = \sum_i \sum_{\tau_i \in T_i} z_{is}^l(t_{-is}^r, \tau_i)[r f_i(\tau_i|t^r)].$$

Hence,

$$z_{is}^l(t^r) f_i(t_i|t^r) = z_{is}^l(t_{-is}^r, t_i) f_i(t_i|t^r) \leq \sum_i w_i^l. \tag{73}$$

It follows that

$$\begin{aligned}
\sum_{l=1}^{L} \mu_{is}^l(t^r) &= \sum_{l=1}^{L} \bar{q}_k^r z_{is}^l(t^r) \\
&\leq \sum_{l=1}^{L} K^l z_{is}^l(t^r) \quad \text{(Applying Step 1)} \\
&\leq L \max_l [K^l z_{is}^l(t^r)] \\
&= L \max_l [K^l \frac{z_{is}^l(t^r)}{f_i(t_i|t^r)} f_i(t_i|t^r)] \\
&\leq L \max_l [K^l \frac{\sum_i w_i^l}{f_i(t_i|t^r)}] \quad \text{(By (73))} \\
&\leq L \max_l [K^l \frac{\sum_i w_i^l}{\rho_i(t_i|\theta) - \gamma}] \quad \text{(By (72))} \\
&< w_i^{L+1} \quad \text{(By (70))}.
\end{aligned}$$

Step 7:[13] For any $\rho > 0$, there exists an $\hat{r} > 0$ such that for any $r > \hat{r}$, any $(i,s) \in N_r$, any $t_i \in T_i$, any $t_i' \in T_i$ and any $\delta_i : \mathbb{R}^{L+1} \to A_i$,

$$\sum_{t^r_{-is}} \sum_{\theta} [u_i(x_{is}[\mu(t^r_{-is}, t_i))], \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i'), \delta_i(\pi(\mu(t^r_{-is}, t_i')))], \theta)] P^r(\theta, t^r_{-is}|t_i) \geq -(12K_1 + 2)\rho.$$

In particular, we want to show the following:

$$\sum_{t^r_{-is}} \sum_{\theta} [u_i(x_{is}[\mu(t^r_{-is}, t_i)], \theta) - u_i(x_{is}[\mu_{-is}(t^r), \delta_i(\pi(\mu(t^r_{-is}, t_i)))], \theta)] P^r(\theta, t^r_{-is}|t_i) + \sum_{t^r_{-is}} \sum_{\theta} [u_i(x_{is}[\mu_{-is}(t^r), \delta_i(\pi(\mu(t^r_{-is}, t_i)))], \theta)$$
$$-u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i'), \delta_i(\pi(\mu(t^r_{-is}, t_i')))], \theta)] P^r(\theta, t^r_{-is}|t_i) \geq -(12K_1 + 3)\rho. \tag{74}$$

To see this, first choose $\rho > 0$. Pick $\hat{r} > 0$ which satisfies the hypotheses of all previous steps. By the claim in step 4, we define the highest possible utility any agent can get in the compact set as:

$$K_1 = \max_{\theta} \max_{i} \{u_i(2Me; \theta)\}$$

where $e = (1, 1, \cdots, 1) \in \mathbb{R}^{L+1}$. Next we divide the proof in the following two sub-steps.

Step 7.1: First, define

$$Q(t_i) = \{q(t^r_{-is}, t_i) | (t^r_{-is}, t_i) \in T^r \setminus B^r_0\}.$$

For any $q \in Q(t_i)$, by condition (ii) in the definition of REE, there exists a function $\hat{z}_{is}(q, t_i)$ such that

$$z^r_{is}(t^r_{-is}, t_i) = \hat{z}_{is}(q, t_i). \tag{75}$$

for all t^r_{-is} satisfying $q(t^r_{-is}, t_i) = q$. Now we consider the first part of (76) which is only about choosing a different action.

$$\sum_{t^r_{-is}} \sum_{\theta} [u_i(x_{is}[\mu(t^r_{-is}, t_i)], \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\pi(\mu(t^r_{-is}, t_i)))], \theta)] P^r(\theta, t^r_{-is}|t_i)$$

(By (49) and $\pi(\mu(t^r_{-is}, t_i)) = \bar{q}^r_k$ if $(t^r_{-is}, t_i) \in B^r_k$.)

$$\geq \sum_{k} \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_i) \in B^r_k}} \sum_{\theta} [u_i(x_{is}[\mu(t^r_{-is}, t_i)], \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\bar{q}^r_k)], \theta)] P^r(\theta, t^r_{-is}|t_i) - 2K_1\rho$$

(By (63))

$$\geq \sum_{k} \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_i) \in B^r_k}} \sum_{\theta} [u_i(z^r_{is}(t^r_{-is}, t_i), \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\bar{q}^r_k)], \theta)] P^r(\theta, t^r_{-is}|t_i) - 2K_1\rho - \rho$$

(By (75))

$$= \sum_{q \in Q(t_i)} \sum_{k} \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_i) \in B^r_k \\ q(t^r_{-is}, t_i) = q}} \sum_{\theta} [u_i(\hat{z}_{is}(q, t_i), \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\bar{q}^r_k)], \theta)] P^r(\theta, t^r_{-is}|t_i) - 2K_1\rho - \rho$$

$$= \sum_{q \in Q(t_i)} \sum_{k} \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_i) \in B^r_k \\ q(t^r_{-is}, t_i) = q}} \sum_{\theta} [u_i(\hat{z}_{is}(q, t_i), \theta) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q], \theta)] P^r(\theta, t^r_{-is}|t_i)$$

$$+ \sum_{q \in Q(t_i)} \sum_{k} \sum_{\substack{t^r_{-is}: \\ (t^r_{-is}, t_i) \in B^r_k \\ q(t^r_{-is}, t_i) = q}} \sum_{\theta} [u_i(y_{is}[\delta_i(\bar{q}^r_k)|q], \theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is}, t_i), \delta_i(\bar{q}^r_k)], \theta)] P^r(\theta, t^r_{-is}|t_i) - 2K_1\rho - \rho$$

[13] In this step, we suppress the dependency of B^r_k on γ to simplify notations.

(By (64))

$$\geq \sum_{q\in Q(t_i)} \sum_k \sum_{\substack{t^r_{-is}:\\ (t^r_{-is},t_i)\in B^r_k\\ q(t^r_{-is},t_i)=q}} \sum_\theta [u_i(\hat{z}_{is}(q,t_i),\theta) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q],\theta)]P^r(\theta,t^r_{-is}|t_i) - 2K_1\rho - 2\rho$$

$$= \sum_{q\in Q(t_i)} \sum_k \sum_{\substack{t^r_{-is}:\\ (t^r_{-is},t_i)\in B^r_k\\ q(t^r_{-is},t_i)=q}} \sum_\theta [u_i(\hat{z}_{is}(q,t_i),\theta) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q],\theta)]P^r(\theta|t^r_{-is},t_i)P^r(t^r_{-is}|t_i) - 2K_1\rho - 2\rho$$

(By (51))

$$\geq \sum_{q\in Q(t_i)} \sum_k \sum_{\substack{t^r_{-is}:\\ (t^r_{-is},t_i)\in B^r_k\\ q(t^r_{-is},t_i)=q}} [u_i(\hat{z}_{is}(q,t_i),\theta_k) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q],\theta_k)]P^r(t^r_{-is}|t_i) - 4K_1\rho - 2\rho$$

$$= \sum_{q\in Q(t_i)} \sum_k [u_i(\hat{z}_{is}(q,t_i),\theta_k) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q],\theta_k)] \sum_{\substack{t^r_{-is}:\\ (t^r_{-is},t_i)\in B^r_k\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i) - 4K_1\rho - 2\rho$$

$$= \sum_{q\in Q(t_i)} \sum_k [u_i(\hat{z}_{is}(q,t_i),\theta_k) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q],\theta_k)] \frac{\sum_{\substack{t^r_{-is}:\\ (t^r_{-is},t_i)\in B^r_k\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i)}{\sum_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i)} \sum_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i) - 4K_1\rho - 2\rho$$

(By (52))

$$\geq \sum_{q\in Q(t_i)} \sum_k [u_i(\hat{z}_{is}(q,t_i),\theta_k) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q],\theta_k)] \frac{\sum_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(\theta_k,t^r_{-is}|t_i)}{\sum_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i)} \sum_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i) - 6K_1\rho - 2\rho$$

$$= \sum_{q\in Q(t_i)} \{\sum_k [u_i(\hat{z}_{is}(q,t_i),\theta_k) - u_i(y_{is}[\delta_i(\bar{q}^r_k)|q],\theta_k)]P^r(\theta_k|q,t_i)\} \sum_{\substack{t^r_{-is}:\\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i) - 6K_1\rho - 2\rho$$

(By the definition of REE and $x_{is}[\delta_i(\bar{q}^r_k)|q] \in \beta_i(q)$)

$$\geq -6K_1\rho - 2\rho.$$

Step 7.2: Now we consider the second part of (76) which is only about misreporting.

$$\sum_{t^r_{-is}} \sum_\theta u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(\pi(\mu(t^r_{-is},t_i)))],\theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(\pi(\mu(t^r_{-is},t'_i)))],\theta)]P^r(\theta,t^r_{-is}|t_i)$$

(By (49))

$$\geq \sum_k \sum_{\substack{t^r_{-is}:\\ (t^r_{-is},t_i)\in B^r_k}} \sum_\theta [u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(q(t^r_{-is},t_i))],\theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(q(t^r_{-is},t'_i))],\theta)]P^r(\theta,t^r_{-is}|t_i) - 2K_1\rho$$

$$= \sum_k \sum_{\substack{t^r_{-is}\\ :(t^r_{-is},t_i)\in B^r_k\\ (t^r_{-is},t'_i)\notin B^r_k\cup B^r_0}} \sum_\theta [u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(q(t^r_{-is},t_i))],\theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(q(t^r_{-is},t'_i))],\theta)]P^r(\theta,t^r_{-is}|t_i)$$

$$+\sum_{k}\sum_{\substack{t^r_{-is}\\:(t^r_{-is},t_i)\in B^r_k\\(t^r_{-is},t'_i)\in B^r_k}}\sum_{\theta}[u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(q(t^r_{-is},t_i))],\theta)-u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(q(t^r_{-is},t'_i))],\theta)]P^r(\theta,t^r_{-is}|t_i)$$

$$+\sum_{k}\sum_{\substack{t^r_{-is}\\:(t^r_{-is},t_i)\in B^r_k\\(t^r_{-is},t'_i)\in B^r_0}}\sum_{\theta}[u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(q(t^r_{-is},t_i))],\theta)-u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(q(t^r_{-is},t'_i))],\theta)]P^r(\theta,t^r_{-is}|t_i)-2K_1\rho$$

(By (53))

$$\geq\sum_{k}\sum_{\substack{t^r_{-is}\\:(t^r_{-is},t_i)\in B^r_k\\(t^r_{-is},t'_i)\notin B^r_k\cup B^r_0}}\sum_{\theta}[u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(q(t^r_{-is},t_i))],\theta)-u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(q(t^r_{-is},t'_i))],\theta)]P^r(\theta,t^r_{-is}|t_i)$$

$$+\sum_{k}\sum_{\substack{t^r_{-is}\\:(t^r_{-is},t_i)\in B^r_k\\(t^r_{-is},t'_i)\in B^r_k}}\sum_{\theta}[u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(q(t^r_{-is},t_i))],\theta)-u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(q(t^r_{-is},t'_i))],\theta)]P^r(\theta,t^r_{-is}|t_i)-4K_1\rho$$

(By (50))

$$\geq\sum_{k}\sum_{\substack{t^r_{-is}\\:(t^r_{-is},t_i)\in B^r_k\\(t^r_{-is},t'_i)\in B^r_k}}\sum_{\theta}[u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(\bar{q}^r_k)],\theta)-u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(\bar{q}^r_k)],\theta)]P^r(\theta,t^r_{-is}|t_i)-6K_1\rho$$

(By (65))

$$\geq-\rho-6K_1\rho.$$

To sum up, we have

$$\sum_{t^r_{-is}}\sum_{\theta}[u_i(x_{is}[\mu(t^r)],\theta)-u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(\pi(\mu(t^r_{-is},t'_i)))],\theta)]P^r(\theta,t^r_{-is}|t_i)\geq-(12K_1+3)\rho.$$

Step 8: In this step, we will show that for any $\varepsilon>0$ there exists a $\hat{r}>0$ such that for any $r>\hat{r}$ the constructed mechanism μ is a type symmetric $\varepsilon-$ICSFM. Choose $\varepsilon>0$. By step 6, choose $\gamma\leq\varepsilon$ and let $S^r=\cup_{k\in J_m}B^r_k(\gamma)$. Then according to (71) we have

$$Prob\{\tilde{t}^r\in S^r\}=Prob\{\tilde{t}^r\in\cup_{k\in J_m}B^r_k(\gamma)\}\geq1-\gamma\geq1-\varepsilon.$$

Therefore, (10) holds. Moreover, we know that by step 6 the constructed mechanism is feasible for any $t^r\in S^r$. Choose $\rho=\frac{\varepsilon}{12K_1+3}$. Then step 7 implies that there exists a $\hat{r}>0$ such that for any $r>\hat{r}$,

$$\sum_{t^r_{-is}}\sum_{\theta}[u_i(x_{is}[\mu(t^r)],\theta)-u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(\pi(\mu(t^r_{-is},t'_i)))],\theta)]P^r(\theta,t^r_{-is}|t_i)\geq-\varepsilon.$$

Appendix D: Proof of Theorem 3

The proof of Theorem 3 is almost identical to the proof of Theorem 2. The only differences appear in step 4 where we construct the mechanism and step 7 where we show that the constructed mechanism is an approximate ICSFM. Therefore, in this section, we will merely write down the differences and the reader could substitute them into the proof of Theorem 2 as the proof of Theorem 3.

Step 4: In this step, we construct the mechanism $\mu:T^r\to A^r$ as follows: for each (i,s), each $t^r\in T^r$ and each $l\neq L+1$,

$$\mu^l_{is}(t^r)=q^l(t^r)z^l_{is}(t^r).$$

Note that the price determined by the mechanism is the following:

$$\pi^l(\mu(t^r)) = \frac{\sum_{i=1}^{n}\sum_{s=1}^{r}\mu_{is}^l(t^r)}{r\sum_{i=1}^{n} w_i^l} = \frac{q^l(t^r)\sum_{s=1}^{r}\sum_{i=1}^{n} z_{is}^l(t^r)}{r\sum_{i=1}^{n} w_i^l} = q^l(t^r).$$

and the corresponding allocation is

$$\begin{aligned}
x_{is}^l[\mu(t^r)] &= \frac{\mu_{is}^l(t^r)}{\pi^l(\mu^r(t^r))} = \frac{q^l(t^r)z_{is}^l(t^r)}{q^l(t^r)} = z_{is}^l(t^r) \quad \text{if } l \neq L+1; \\
x_{is}^{L+1}[\mu(t^r)] &= w_i^{L+1} + \sum_{l=1}^{L}\pi^l(\mu(t^r))w_i^l - \sum_{l=1}^{L}\mu_{is}^l(t^r) \\
&= w_i^{L+1} + \sum_{l=1}^{L} q^l(t^r)w_i^l - \sum_{l=1}^{L} q^l(t^r)z_{is}^l(t^r) \\
&= z_{is}^{L+1}(t^r).
\end{aligned}$$

Step 7:[14] For any $\rho > 0$, there exists an $\hat{r} > 0$ such that for any $r > \hat{r}$, any $(i,s) \in N_r$, any $t_i \in T_i$, any $t_i' \in T_i$ and any $\delta_i : \mathbb{R}^{L+1} \to A_i$,

$$\sum_{t_{-is}^r}\sum_{\theta}[u_i(x_{is}[\mu(t_{-is}^r, t_i))], \theta) - u_i(x_{is}[\mu_{-is}(t_{-is}^r, t_i'), \delta_i(\pi(\mu(t_{-is}^r, t_i')))], \theta)]P^r(\theta, t_{-is}^r|t_i) \geq -(12K_1+3)\rho.$$

In particular, we want to show the following:

$$\begin{aligned}
&\sum_{t_{-is}^r}\sum_{\theta}[u_i(x_{is}[\mu(t_{-is}^r, t_i)], \theta) - u_i(x_{is}[\mu_{-is}(t_{-is}^r, t_i), \delta_i(\pi(\mu(t_{-is}^r, t_i')))], \theta)]P^r(\theta, t_{-is}^r|t_i) \\
+&\sum_{t_{-is}^r}\sum_{\theta}[u_i(x_{is}[\mu_{-is}(t_{-is}^r, t_i), \delta_i(\pi(\mu(t_{-is}^r, t_i')))], \theta) - u_i(x_{is}[\mu_{-is}(t_{-is}^r, t_i'), \delta_i(\pi(\mu(t_{-is}^r, t_i')))], \theta)]P^r(\theta, t_{-is}^r|t_i) \geq -(12K_1+3)\rho.
\end{aligned} \tag{76}$$

To see this, first choose $\rho > 0$. Pick $\hat{r} > 0$ which satisfies the hypotheses of all previous steps. In particular, by the continuity of δ_i and assumption A.3, we have that for each $k \in J_m$ there exists a $\bar{t}^{r,k}$ such that for any $t^r \in B_k^r$,

$$||u_i(x_{is}[\mu_{-is}(t_{-is}^r, t_i), \delta_i(q(\bar{t}^{r,k}))], \theta_k) - u_i(x_{is}[\mu_{-is}(t_{-is}^r, t_i), \delta_i(q(t^r))], \theta_k)|| \geq -\rho. \tag{77}$$

Next we divide the proof in the following two sub-steps.

Step 7.1: First consider the first part of (76) which is only about choosing a different action.

$$\begin{aligned}
&\sum_{t_{-is}^r}\sum_{\theta}[u_i(x_{is}[\mu(t_{-is}^r, t_i)], \theta) - u_i(x_{is}[\mu_{-is}(t_{-is}^r, t_i), \delta_i(\pi(\mu(t_{-is}^r, t_i')))], \theta)]P^r(\theta, t_{-is}^r|t_i) \\
&\geq \sum_{q \in Q(t_i)}\sum_{k}\sum_{\substack{t_{-is}^r: \\ (t_{-is}^r, t_i) \in B_k^r \\ q(t_{-is}^r, t_i) = q}} [u_i(\hat{z}_{is}(q, t_i), \theta_k) - u_i(y_{is}[\delta_i(q(t_{-is}^r, t_i'))|q], \theta_k)]P^r(t_{-is}^r|t_i) - 4K_1\rho - \rho \\
&\qquad\qquad \text{(By (77))} \\
&\geq \sum_{q \in Q(t_i)}\sum_{k}\sum_{\substack{t_{-is}^r: \\ (t_{-is}^r, t_i) \in B_k^r \\ q(t_{-is}^r, t_i) = q}} [u_i(\hat{z}_{is}(q, t_i), \theta_k) - u_i(y_{is}[\delta_i(q(\bar{t}^{r,k}))|q], \theta_k)]P^r(t_{-is}^r|t_i) - 4K_1\rho - 2\rho
\end{aligned}$$

[14] In this step, we suppress the dependency of B_k^r on γ to simplify notations.

$$= \sum_{q \in Q(t_i)} \sum_k [u_i(\hat{z}_{is}(q,t_i),\theta_k) - u_i(y_{is}[\delta_i(q(\bar{t}^{r,k}))|q],\theta_k)] \sum_{\substack{t^r_{-is}: \\ (t^r_{-is},t_i)\in B^r_k \\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i) - 4K_1\rho - 2\rho$$

$$\geq \sum_{q \in Q(t_i)} \{\sum_k [u_i(\hat{z}_{is}(q,t_i),\theta_k) - u_i(y_{is}[\delta_i(q(\bar{t}^{r,k}))|q],\theta_k)] P^r(\theta_k|q,t_i)\} \sum_{\substack{t^r_{-is}: \\ q(t^r_{-is},t_i)=q}} P^r(t^r_{-is}|t_i) - 6K_1\rho - 2\rho$$

(By the definition of REE and $x_{is}[\delta_i(q(\bar{t}^{r,k}))|q] \in \beta_i(q)$)

$\geq -6K_1\rho - 2\rho$.

Step 7.2: Now we consider the second part of (76) which is only about misreporting. Following the same steps in the proof of Theorem 2, we have

$$\sum_{t^r_{-is}} \sum_{\theta} u_i(x_{is}[\mu_{-is}(t^r_{-is},t_i),\delta_i(\pi(\mu(t^r_{-is},t'_i)))],\theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(\pi(\mu(t^r_{-is},t'_i)))],\theta)]P^r(\theta,t^r_{-is}|t_i) \geq -\rho - 6K_1\rho.$$

To sum up, we have

$$\sum_{t^r_{-is}} \sum_{\theta} [u_i(x_{is}[\mu(t^r)],\theta) - u_i(x_{is}[\mu_{-is}(t^r_{-is},t'_i),\delta_i(\pi(\mu(t^r_{-is},t'_i)))],\theta)]P^r(\theta,t^r_{-is}|t_i) \geq -(12K_1+3)\rho.$$

References

Codognato, G. and Ghosal, S. (2003). Self-fulfilling mechanisms and rational expectations in large markets. *Journal of Mathematical Economics*, 39(5-6):421–431.

Dubey, P., Geanakoplos, J., and Shubik, M. (1987). The revelation of information in strategic market games: A critique of rational expectations equilibrium. *Journal of Mathematical Economics*, 16(2):105–137.

Forges, F. and Minelli, E. (1997). Self-fulfilling mechanisms and rational expectations. *Journal of Economic Theory*, 75(2):388–406.

Grossman, S. J. and Stiglitz, J. E. (1980). On the impossibility of informationally efficient markets. *The American economic review*, 70(3):393–408.

McLean, R. and Postlewaite, A. (2002). Informational size and incentive compatibility. *Econometrica*, 70(6):2421–2453.

McLean, R. and Postlewaite, A. (2004). Informational size and efficient auctions. *The Review of Economic Studies*, 71(3):809–827.

McLean, R. and Postlewaite, A. (2005). Core convergence with asymmetric information. *Games and Economic Behavior*, 50(1):58–78.

McLean, R. P. and Postlewaite, A. (2017). A dynamic non-direct implementation mechanism for interdependent value problems. *Games and Economic Behavior*, 101:34–48.

Milgrom, P. R. (1981). Rational expectations, information acquisition, and competitive bidding. *Econometrica: Journal of the Econometric Society*, pages 921–943.

Postlewaite, A. and Schmeidler, D. (1978). Approximate efficiency of non-walrasian nash equilibria. *Econometrica: journal of the Econometric Society*, pages 127–135.

Radner, R. (1979). Rational expectations equilibrium: Generic existence and the information revealed by prices. *Econometrica: Journal of the Econometric Society*, pages 655–678.

Shapley, L. and Shubik, M. (1977). Trade using one commodity as a means of payment. *Journal of political economy*, 85(5):937–968.

Sun, Y. (2006). The exact law of large numbers via fubini extension and characterization of insurable risks. *Journal of Economic Theory*, 126(1):31–69.